BINOMIAL SKEW POLYNOMIAL RINGS, ARTIN-SCHELTER REGULARITY, AND BINOMIAL SOLUTIONS OF THE YANG-BAXTER EQUATION

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ABSTRACT. Let $\mathbf{k}$ be a field and X be a set of n elements. We introduce and study a class of quadratic $\mathbf{k}$-algebras called *quantum binomial algebras*. Our main result shows that such an algebra A defines a solution of the classical Yang-Baxter equation (YBE), if and only if its Koszul dual $A^!$ is Frobenius of dimension n, with a *regular socle* and for each $x, y \in X$ an equality of the type $xyy = \alpha zzt$, where $\alpha \in k \setminus \{0\}$, and $z, t \in X$ is satisfied in A. We prove the equivalence of the notions *a binomial skew polynomial ring* and *a binomial solution of YBE*. This implies that the Yang-Baxter algebra of such a solution is of Poincaré-Birkhoff-Witt type, and possesses a number of other nice properties such as being Koszul, Noetherian, and an Artin-Schelter regular domain.

1. INTRODUCTION

In the paper we work with associative finitely presented graded $\mathbf{k}$-algebras $A = \bigoplus_{i\geq 0} A_i$, where $\mathbf{k}$ is a field, $A_0 = \mathbf{k}$, and A is generated by A_1. We restrict our attention to a class of algebras with quadratic binomial defining relations and study the close relations between different algebraic notions such as *Artin-Schelter regular rings*, *Yang-Baxter algebras* defined via *binomial solutions* of the classical Yang-Baxter equation, and a class of quadratic standard finitely presented algebras with a Poincaré-Birkhoff-Witt type $\mathbf{k}$-basis, called *binomial skew polynomial rings*.

Following a classical tradition (and recent trend), we take a combinatorial approach to study A. The properties of A will be read off a presentation $A = \mathbf{k}\langle X\rangle/(\Re)$, where X is a finite set of indeterminates of degree 1, $\mathbf{k}\langle X\rangle$ is the unitary free associative algebra generated by X, and $(\Re)$ is the two-sided ideal of relations, generated by a *finite* set $\Re$ of homogeneous polynomials.

Artin and Schelter [3] call a graded algebra A *regular* if

(i) A has *finite global dimension* d, that is, each graded A-module has a free resolution of length at most d.
(ii) A has *finite Gelfand-Kirillov dimension*, meaning that the integer-valued function $i \mapsto \dim_{\mathbf{k}} A_i$ is bounded by a polynomial in i.
(iii) A is *Gorenstein*, that is, $Ext^i_A(\mathbf{k}, A) = 0$ for $i \neq d$ and $Ext^d_A(\mathbf{k}, A) \cong \mathbf{k}$.

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The regular rings were introduced and studied first in [3]. When $d \leq 3$ all regular algebras are classified. The problem of classification of regular rings is difficult and remains open even for regular rings of global dimension 4. The study of Artin-Schelter regular rings, their classification, and finding new classes of such rings is one of the basic problems for noncommutative geometry. Numerous works on this topic appeared during the last 16 years, see for example [4], [20], [21], [28], [30], [31], etc.

For the rest of the paper we fix X. If an enumeration $X = \{x_1, \dots, x_n\}$ is given, we will consider the degree-lexicographic order $\prec$ on $\langle X\rangle$, the unitary free semigroup generated by X (we assume $x_1 \prec x_2 \prec \cdots \prec x_n$).

Suppose the algebra A is given with a finite presentation $A = \mathbf{k}\langle x_1, \dots, x_n\rangle/(\Re)$.

In some cases we will ignore a given enumeration on X and will search for an appropriate enumeration (if any), which provides a degree-lexicographic ordering $\prec$ with respect to which the relations $\Re$ become of *skew-polynomial type*, see Definition 1.7.

Recall that a monomial $u \in \langle X\rangle$ is *normal* mod $\Re$ (with respect to the chosen order) if u does not contain as a segment any of the highest monomials of the polynomials in $\Re$. By $N(\Re)$ we denote the set of all normal mod $\Re$ monomials.

Notation 1.1. As usual, we denote $\mathbf{k}^{\times} = \mathbf{k} \setminus \{0\}$. If $\omega = x_{i_1} \cdots x_{i_m} \in \langle X\rangle$, its length m is denoted by $\mid \omega \mid$. X^m will denote the set of all words of length m in the free semigroup $\langle X\rangle$. We shall identify the m-th tensor power $V^{\otimes m}$ with $V^m = Span_{\mathbf{k}} X^m$, the $\mathbf{k}$-vector space spanned by all monomials of length m.

We shall introduce now a class of quadratic algebras with binomial relations, we call them *quantum binomial algebras*, which contains various algebras, such as binomial skew polynomial rings, [9], [10], [11], the Yang-Baxter algebras defined via the so called *binomial solutions* of the classical Yang-Baxter equation, [14], the semigroup algebras of semigroups of skew type, [15], etc. all of which are actively studied.

Definition 1.2. Let $A(\mathbf{k}, X, \Re) = \mathbf{k}\langle X\rangle/(\Re)$ be a finitely presented $\mathbf{k}$-algebra with a set of generators X consisting of n elements, and quadratic defining relations $\Re \subset \mathbf{k}\langle X\rangle$. The relations $\Re$ are called *quantum binomial relations* and A is *a quantum binomial algebra* if the following conditions hold.

(a) Each relation in $\Re$ is of the shape $xy - c_{yx}y'x'$, where $x, y, x', y' \in X$, and $c_{xy} \in \mathbf{k}^{\times}$ (this is what we call *a binomial relation*).
(b) Each $xy, x \neq y$ of length 2 occurs at most once in $\Re$.
(c) Each relation is *square-free*, i.e. it does not contain a monomial of the shape xx, $x \in X$.
(d) The relations $\Re$ are *non degenerate*, i.e. the canonical bijection $r = r(\Re) : X^2 \longrightarrow X^2$, associated with $\Re$, see Definition 1.3 is left and right non degenerate.

A quantum binomial algebra A is called *standard quantum binomial algebra* if the set $\Re$ is a Gröbner basis with respect to the degree-lexicographic ordering $\prec$, where some appropriate enumeration of X is chosen, $X = \{x_1 \prec x_2 \prec \cdots \prec x_n\}$.

Definition 1.3. Let $\Re \subset \mathbf{k}\langle X\rangle$ be a set of quadratic binomial relations, satisfying conditions (a) and (b) of Definition 1.2. *The automorphism associated with* $\Re$, $R = R(\Re) : V^2 \longrightarrow V^2$, is defined as follows: on monomials which occur in some

relation, $xy - c_{xy}y'x' \in \Re$, we set

$$R(xy) = c_{xy}y'x', \quad \text{and} \quad R(y'x') = (c_{xy})^{-1}xy.$$

If xy, does not occur in any relation ($x = y$ is also possible), then we set $R(xy) = xy$.

We also define a bijection $r = r(\Re) : X^2 \longrightarrow X^2$ as $r(xy) = y'x'$, and $r(y'x') = xy$, if $xy - c_{xy}y'x' \in \Re$. If xy, does not occur in any relation then we set $r(xy) = xy$. We call $r(\Re)$ *the (set-theoretic) canonical map associated with* $\Re$.

We say that r is *nondegenerate*, if the two maps $\mathcal{L}_x : X \longrightarrow X$, and $\mathcal{R}_y : X \longrightarrow X$ determined via the formula:

$$r(xy) = \mathcal{L}_x(y)\mathcal{R}_y(x)$$

are bijections for each $x, y \in X$.

R is called *non-degenerate* if r is non-degenerate. In this case we shall also say that the defining relations $\Re$ are *non degenerate binomial relations.*

Definition 1.4. With each quantum binomial set of relations $\Re$ we associate a set of semigroup relations $\Re_0$, obtained by setting $c_{xy} = 1$, for each relation $(xy - c_{yx}y'x') \in \Re$. In other words,

$$\Re_0 = \{xy = y'x' \mid xy - c_{xy}y'x' \in \Re\}$$

The *semigroup associated to* $A(\mathbf{k}, X, \Re)$ is $\mathcal{S}_0 = \mathcal{S}_0(X, \Re_0) = \langle X; \Re_0\rangle$, we also refer to it as *quantum binomial semigroup.* The semigroup algebra associated to $A(\mathbf{k}, X, \Re)$ is $\mathcal{A}_0 = \mathbf{k}\langle X\rangle/(\Re_0)$, which is isomorphic to $\mathbf{k}\mathcal{S}_0$.

The following lemma gives more precise description of the relations in a quantum binomial algebra. We give the proof in Section 2.

Lemma 1.5. *Let $A(\boldsymbol{k}, X, \Re)$ be a quantum binomial algebra, let $\mathcal{S}_0$ be the associated quantum binomial semigroup. Then the following conditions are satisfied.*

(i) *$\Re$ contains precisely $\binom{n}{2}$ relations*
(ii) *Each monomial $xy \in X^2, x \neq y$, occurs exactly once in $\Re$.*
(iii) *$xy - c_{yx}y'x' \in \Re$, implies $y' \neq x, x' \neq y$.*
(iv) *The left and right Ore conditions, (see Definition 2.4) are satisfied in $\mathcal{S}_0$.*

Clearly, if the set $\Re$ is a Gröbner basis then $\Re_0$ is also a Gröbner basis. Therefore, for a standard quantum binomial algebra $A(\mathbf{k}, X, \Re)$ the associated semigroup algebra $\mathcal{A}_0$ is also standard quantum binomial.

Example 1.6. a) Each binomial skew polynomial ring, see Definition 1.7 is a standard quantum binomial algebra.

b) Let R be a binomial solution of the classical Yang-Baxter equation, see Definition 1.12, and let $\Re(R)$ be the corresponding quadratic relations, then the Yang-Baxter algebra $A = \mathbf{k}\langle X\rangle/(\Re)$ is a quantum binomial algebra.

c) $A = k\langle x_1, x_2, x_3, x_4\rangle/(x_4x_3 - x_2x_4, x_4x_2 - x_1x_3, x_4x_1 - x_3x_4, x_3x_2 - x_2x_3, x_3x_1 - x_1x_4, x_2x_1 - x_1x_2)$ is a quantum binomial algebra, which is not standard quantum binomial, i.e. whatever enumeration on X we fix, the set of relations $\Re$ is not a Gröbner basis with respect to $\prec$. This can be deduced by direct computations, but one needs to check all possible, 4! enumerations of X, which is too long. (In particular if we chose the given enumeration, the ambiguity $x_4x_3x_1$ is not solvable). Here we give another proof, which is universal and does not depend on the enumeration.

Assume, on the contrary, $\Re$ is a Gröbner basis, with respect to an appropriate enumeration. Therefore A is a binomial skew polynomial ring and the cyclic condition is satisfied, see Definition 1.14. Now the relations $x_4x_3 - x_2x_4, x_4x_2 - x_1x_3$, give a contradiction.

We single out an important subclass of standard quantum binomial algebras with a Poincaré-Birkhoff-Witt type $\mathbf{k}$-basis, namely *the binomial skew polynomial rings.* These rings were introduced and studied in [9], [10], [11], [16], [19]. Laffaille calls them *quantum binomial algebras.* He shows in [19], that for $\mid X \mid \leq 6$, the associated automorphism R is a solution of the Yang-Baxter equation. We prefer to keep the name "binomial skew polynomial rings" since we have been using this name for already 10 years. It was proven in 1995, see [11] and [16] that the binomial skew polynomial rings provide a new (at that time) class of Artin-Schelter regular rings of global dimension n, where n is the number of generators X. We recall now the definition.

Definition 1.7. [10] A *binomial skew polynomial ring* is a graded algebra $A = \mathbf{k}\langle x_1, \cdots, x_n\rangle/(\Re)$ in which the indeterminates x_i have degree 1, and which has precisely $\binom{n}{2}$ defining relations $\Re = \{x_jx_i - c_{ij}x_{i'}x_{j'}\}_{1\leq i<j\leq n}$ such that

(a) $c_{ij} \in \mathbf{k}^{\times}$;
(b) For every pair i, j $1 \leq i < j \leq n$, the relation $x_jx_i - c_{ij}x_{i'}x_{j'} \in \Re$, satisfies $j > i'$, $i' \leq j'$;
(c) Every ordered monomial x_ix_j, with $1 \leq i < j \leq n$ occurs in the right hand side of some relation in $\Re$;
(d) $\Re$ is the *reduced Gröbner basis* of the two-sided ideal $(\Re)$,(with respect to the order $\prec$ on $\langle X\rangle$) or equivallently the ambiguities $x_kx_jx_i$, with $k > j > i$ do not give rise to new relations in A.

We call $\Re$ *relations of skew-polynomial type* if conditions 1.7 (a), (b) and (c) are satisfied (we do not assume (d)) .

By [5] condition 1.7 (d) may be rephrased by saying that *the set of ordered monomials*
$$\mathcal{N}_0 = \{x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha_n \geq 0 \text{ for } 1 \leq i \leq n\}$$
is a $\mathbf{k}$-basis of A.

Definition 1.8. We say that the semigroup $\mathcal{S}_0$ is *a semigroup of skew-polynomial type*, (or shortly, *a skew-polynomial semigroup*) if it has a standard finite presentation as $\mathcal{S}_0 = \langle X; \Re_0\rangle$, where the set of generators X is ordered: $x_1 \prec x_2 \prec \cdots \prec x_n$, and the set
$$\Re_0 = \{x_jx_i = x_{i'}x_{j'}) \mid 1 \leq i < j \leq n, 1 \leq i' < j' \leq n\},$$
contains precisely $\binom{n}{2}$ quadratic square-free binomial defining relations, each of them satisfying the following conditions:

(i) each monomial $xy \in X^2$, with $x \neq y$, occurs in exactly one relation in $\Re_0$; a monomial of the type xx does not occur in any relation in $\Re_0$;
(ii) if $(x_jx_i = x_{i'}x_{j'}) \in \Re_0$, with $1 \leq i < j \leq n$, then $i' < j'$, and $j > i'$. (further studies show that this also implies $i < j'$ see [10])
(iii) the monomials $x_kx_jx_i$ with $k > j > i, 1 \leq i, j, k, \leq n$ do not give rise to new relations in S_0, or equivalently, cf. [5], $\Re_0$ is a Gröbner basis with respect to the degree-lexicographic ordering of the free semigroup $\langle X\rangle$.

Example 1.9.

$$A_1 = \mathbf{k}\langle x_1, x_2, x_3\rangle/(\Re_1),$$

where

$$\Re_1 = \{x_3x_2 - x_1x_3,\ x_3x_1 - x_1x_3,\ x_2x_1 - x_1x_2\}.$$

Then $\Re_1$ is a Gröbner basis, but it does not satisfy (c) in Definition 1.7, hence A_1 is not a binomial skew polynomial ring. Respectively, the semigroup $\langle X \mid \Re_0\rangle$ is not a skew-polynomial semigroup. (Here $\Re_0$ are the associated semigroup relations as in Definition 1.4.

Example 1.10. Let

$$A_2 = \mathbf{k}\langle x_1, x_2, x_3, x_4\rangle/(\Re_2),$$

where

$$\Re_2 = \{x_4x_3 - ax_3x_4,\ x_4x_2 - bx_1x_3,\ x_4x_1 - cx_2x_3,\\ x_3x_2 - dx_1x_4,\ x_3x_1 - ex_2x_4,\ x_2x_1 - fx_1x_2\},$$

and the coefficients a, b, c, d, e, f satisfy

$$abcdef \neq 0,\ a^2 = f^2 = be/cd = cd/be, a^4 = f^4 = 1.$$

This is a binomial skew polynomial ring. A_2 is regular and left and right Noetherian domain.

A classification of the binomial skew polynomial rings with 4 generators was given in [9], some of those algebras are isomorphic. A computer programme was used in [19] to find all the families of binomial skew polynomial rings in the case $n \leq 6$, some of the algebras there are also isomprphic. One can also find a classification of the binomial skew polynomial rings with 5 generators and various examples of such rings in 6 generators found independently in [8].

Now we recall the definition of the Yang-Baxter equation.

Let V be a vector space over a field $\mathbf{k}$. A linear automorphism R of $V \otimes V$ is *a solution of the Yang-Baxter equation*, (YBE) if the equality

$$R^{12}R^{23}R^{12} = R^{23}R^{12}R^{23} \tag{1.1}$$

holds in the automorphism group of $V \otimes V \otimes V$, where R^{ij} means R acting on the i-th and j-th component.

In 1990 V. Drinfeld [6] posed the problem of studying the *set-theoretic solutions*of YBE.

Definition 1.11. A bijective map $r : X^2 \longrightarrow X^2$, is called a *set-theoretic solution of the Yang-Baxter equation* (YBE) if the braid relation

$$r^{12}r^{23}r^{12} = r^{23}r^{12}r^{23}$$

holds in X^3, where the two bijective maps $r^{ii+1} : X^3 \longrightarrow X^3$, $1 \leq i \leq 2$ are defined as $r^{12} = r \times Id_X$, and $r^{23} = Id_X \times r$.

We use notation (X, r) for nondegenerate involutive set-theoretic solutions of YBE. (For nondegeneracy, see Definition 1.3).

Each set-theoretic solution r of the Yang-Baxter equation induces an operator R on $V \otimes V$ for the vector space V spanned by X, which is, clearly, a solution of 1.1.

Definition 1.12. ([14], Def. 9.1) Let V be a finite dimensional vector space over a field $\mathbf{k}$ with a $\mathbf{k}$-basis $X = \{x_1, \cdots, x_n\}$. Suppose the linear automorphism $R : V \otimes V \longrightarrow V \otimes V$ is a solution of the Yang-Baxter equation.

We say that R is *a binomial solution of the (classical) Yang-Baxter equation* or shortly *binomial solution* if the following conditions hold:

(1) for every pair $i \neq j, 1 \leq i, j \leq n$,

$$R(x_j \otimes x_i) = c_{ij} x_{i'} \otimes x_{j'}, R(x_{i'} \otimes x_{j'}) = \frac{1}{c_{ij}} x_j \otimes x_i, \text{ where } c_{ij} \in k, c_{ij} \neq 0.$$

(2) R is *non-degenerate*, that is *the associated set-theoretic solution* $(X, r(R))$, is *non-degenerate*, where $r = r(R) : X^2 \longrightarrow X^2$ is defined as

$$r(x_j x_i) = x_{i'} x_{j'}, \ r(x_{i'} x_{j'}) = x_j x_i \text{ if } R(x_j \otimes x_i) = c_{ij} x_{i'} \otimes x_{j'},$$

see 1.3, see also [7], [14].

(3) We call the binomial solution R (respectively the set-theoreric solution (X, r)) *square-free* if $R(x \otimes x) = x \otimes x$, (respectively $r(xx) = xx$) for all $x \in X$

Notation 1.13. By $(\mathbf{k}, X, R)$ we shall denote a square-free binomial solution of the classical Yang-Baxter equation.

Each binomial solution $(\mathbf{k}, X, R)$ defines a quadratic algebra $\mathcal{A}_R = \mathcal{A}(\mathbf{k}, X, R)$, namely *the associated Yang-Baxter algebra*, in the sense of Manin [23], see also [14] . The algebra $\mathcal{A}(k, X, R)$ is generated by X and has quadratic defining relations, $\Re(R)$ determined by R as in (1.2):

$$\Re(R) = \{(x_j x_i - c_{ij} x_{i'} x_{j'}) \mid R(x_j \otimes x_i) = c_{ij} x_{i'} \otimes x_{j'}\} \tag{1.2}$$

Given a set-theoretic solution (X, r), we define the quadratic relations $\Re(r)$, *the associated Yang-Baxter semigroup* $S(X, r)$ and the algebra $A(\mathbf{k}, X, r)$ analogously, see [14].

Definition 1.14. [14] Let $A = k\langle X \rangle / (\Re)$ be a quantum binomial algebra, let $\mathcal{S}_0$ be the associated semigroup. We say that A, respectively $\mathcal{S}_0$ satisfies *the weak cyclic condition* if for any $x, y \in X, x \neq y$ the following relations hold in $\mathcal{S}_0$:

$$(yx = x_1 y_1) \in \Re_0 \text{ implies } (yx_1 = x_2 y_1) \in \Re_0, \ (y_1 x = x_1 y_2) \in \Re_0.$$

for some appropriate $x_2, y_2 \in X$. Or equivalently, for all $x, y \in X$ one has¡.

$$\mathcal{R}_{\mathcal{L}_{y(x)}}(y) = \mathcal{R}_x(y), \ \mathcal{L}_{\mathcal{R}_{x(y)}}(x) = \mathcal{L}_y(x).$$

It is shown in [10] that every binomial skew polynomial ring A satisfies the weak cyclic condition. Furthermore, every Yang-Baxter semigroup $S(X, r)$ associated with a set-theoretic solution (X, r) satisfies the weak cyclic condition, [12] and [14].

Remark 1.15. In fact both A and $S(X, r)$ satisfy a stronger condition which we call *the cyclic condition*, see [10], and [14].

For the main results we need to recall the definitions of the Koszul dual algebra and of a Frobenius algebra.

The Koszul dual $A^!$ is defined in [23, **?**]. One can deduce from there the following presentation of $A^!$ in terms of generators and relations.

Definition 1.16. Suppose $A = \mathbf{k}\langle X\rangle/(\Re)$, is a quantum binomial algebra. *The Koszul dual $A^!$ of A*, [23] is the quadratic algebra,

$$\mathbf{k}\langle \xi_1, \cdots, \xi_n\rangle/(\Re^{\perp}),$$

where the set $\Re^{\perp}$ contains precisely $\binom{n}{2} + n$ relations of the following two types:
a) binomials

$$\xi_j\xi_i + (c_{ij})^{-1}\xi_{i'}\xi_{j'} \in \Re^{\perp},\ \text{whenever}\ x_jx_i - c_{ij}x_{i'}x_{j'} \in \Re,\ 1 \leq i \neq j \leq n;$$

and
b) monomials:

$$(\xi_i)^2 \in \Re^{\perp}, 1 \leq i \leq n.$$

Remark 1.17. [23], (see also [28]) Note that if we set $V = Span_{\mathbf{k}}(x_1, x_2, \cdots, x_n)$ $V^* = Span_{\mathbf{k}}(\xi_1, \xi_2 \cdots, \xi_n)$, and define a bilinear pairing $\langle\ |\ \rangle : V^* \otimes V \longrightarrow \mathbf{k}$ by $\langle \xi_i \mid x_j\rangle = \delta_{ij}$, then the relations $\Re^{\perp}$ generate a subspace in $V^* \otimes V^*$ which is orthogonal to the subspace of $V \otimes V$ generated by $\Re$.

Definition 1.18. [23], [24] A graded algebra $A = \bigoplus_{i\geq 0} A_i$ is called *a Frobenius algebra of dimension d*, (or *a Frobenius quantum space of dimension d*) if

(a) $dim(A_d) = 1$, $A_i = 0$, for $i > d$;
(b) For all $j \geq 0$) the multiplicative map $m : A_j \otimes A_{d-j} \rightarrow A_d$ is a perfect duality (nondegenerate pairing).
A is called *a quantum grassmann algebra* if in addition
(c) $dim_{\mathbf{k}} A_i = \binom{d}{i}$, for $1 \leq i \leq d$

The following two theorems are the main results of the paper.

Theorem A 1.19. *Let $X = \{x_1, \cdots, x_n\}$, let $\prec$ be the degree-lexicographic order on $\langle X\rangle$. Suppose $\mathcal{F} = \boldsymbol{k}\langle X\rangle/(\Re^!)$ is a quadratic graded algebra, which has precisely $\binom{n}{2} + n$ defining relations*

$$\Re^! = \Re \bigcup \Re_1,\ \textit{where}\ \Re_1 = \{x_jx_j\}_{1\leq j\leq n},\ \Re = \{x_jx_i - c_{ij}x_{i'}x_{j'}\}_{1\leq i<j\leq n},$$

and the set $\Re$ is such that:

(a) *$\Re$ are relations of skew-polynomial type with respect to $\prec$ (see Definition 1.7);*
(b) *$\Re$ is a Gröbner basis for the ideal $(\Re)$ in $\boldsymbol{k}\langle X\rangle$.*

(In other words, $A = \boldsymbol{k}\langle X\rangle/(\Re)$ is a binomial skew polynomial ring).
Then

(1) *$\Re^!$ is a Gröbner basis for the ideal $(\Re^!)$ in $\boldsymbol{k}\langle X\rangle$ and the set of monomials*

$$\mathcal{N}^! = \{x_1^{\varepsilon_1}x_2^{\varepsilon_2}\cdots x_n^{\varepsilon_n} \mid 0 \leq \varepsilon_i \leq 1,\ \textit{for all}\ 1 \leq i \leq n\}$$

is a $\boldsymbol{k}$-basis of $\mathcal{F}$.
(2) *$\mathcal{F}$ is Koszul.*
(3) *$\mathcal{F}$ is a Frobenius algebra of dimension n. More precisely, $\mathcal{F}$ is graded (by length),*

$$\mathcal{F} = \bigoplus_{i\geq 0} \mathcal{F}_i,\ \textit{where} \tag{1.3}$$

$$
\begin{aligned}
\mathcal{F}_0 &= \boldsymbol{k},\\
\mathcal{F}_i &= Span_{\boldsymbol{k}}\{u \mid u \in \mathcal{N}^! \text{ and } \mid u \mid = i\},\ \text{for } 1 \leq i \leq n,\\
\mathcal{F}_n &= Span_{\boldsymbol{k}}(W),\ \text{where } W = x_1x_2\cdots x_n,\\
\mathcal{F}_{n+j} &= 0 \text{ for } j \geq 1.
\end{aligned}
$$

(4) *Furthermore, $\mathcal{F}$ is a quantum grassmann algebra:*

$$dim_{\boldsymbol{k}}\mathcal{F}_i = \binom{n}{i},\ \text{for } 1 \leq i \leq n.$$

Theorem B 1.20. *Let $A = \boldsymbol{k}\langle X\rangle/(\Re)$ be a quantum binomial algebra. Then the following three conditions are equivalent.*

(1) *A satisfies the weak cyclic condition. The Koszul dual $A^!$ is Frobenius of dimension n, and has a regular socle, see Definition 2.14.*
(2) *A is a binomial skew polynomial ring, with respect to some appropriate enumeration of X.*
(3) *The automorphism $R = R(\Re) : V^2 \longrightarrow V^2$ is a solution of the classical Yang-Baxter equation, so A is a Yang-Baxter algebra.*

Furthermore, each of these conditions implies that

(a) *There exists an enumeration of X, $X = \{x_1, \cdots, x_n\}$, such that the set of ordered monomials $\mathcal{N}_0$ forms a $\boldsymbol{k}$-basis of A , i.e. A satisfies an analogue of Poincaré-Birkhoff-Witt theorem;*
(b) *A is Koszul;*
(c) *A is left and right Noetherian.*
(d) *A is an Artin-Schelter regular domain.*
(e) *A satisfies a polynomial identity.*
(f) *A is catenary.*

2. The principal monomial and regularity

Conventions 2.1. In this section we assume that $A = A(\mathbf{k}, X, \Re) = \mathbf{k}\langle X\rangle/(\Re)$ is a quantum binomial algebra, $\mathcal{S}_0 = \langle X; \Re_0\rangle$ is the associated quantum binomial semigroup. $R : V^2 \longrightarrow V^2$, and $r : X^2 \longrightarrow X^2$, where $R = R(\Re)$ and $r = r(\Re)$, are the maps associated with $\Re$, defined in 1.3. Furthermore, till the end of the section we shall assume that the Koszul dual $A^!$ is Frobenius.

Remark 2.2. By our assumption

a) $A^!$ is graded by length:

$$A^! = \bigoplus_{0\leq i\leq n} A^!_i,\ \text{where } dim(A^!_n) = 1;$$

and

b) The multiplication function $m : A^!_j \otimes A^!_{n-j} \to A^!_n$ is a non-degenerate pairing, for all $j \geq 0$.

The one dimensional component $A^!_n$ is called *the socle of* $A^!$

Notation 2.3. For $m \geq 2$, $\Delta_m = \{x^m \mid x \in X\}$ denotes the diagonal of X^m.

Definition 2.4. Let $\mathcal{S}_0$ be a semigroup generated by X. a) $\mathcal{S}_0$ satisfies *the right Ore condition* if for every pair $a, b \in X$ there exists a unique pair $x, y \in X$, such that $ax = by$; b) $\mathcal{S}_0$ satisfies *the left Ore condition* if for every pair $a, b \in X$ there exists a unique pair $z, t \in X$, such that $za = tb$.

Proof of lemma. 1.5. Suppose $A(\mathbf{k}, X, \Re)$ is a quantum binomial algebra. By Definition 1.2 the relations in $\Re$ are square-free, therefore $r(xx) = xx$, and $\mathcal{L}_x(x) = x = \mathcal{R}_x(x)$ for every $x \in X$. Suppose $x, y \in X, x \neq y$. The nondegeneracy condition implies

$$\mathcal{L}_x(y) \neq \mathcal{L}_x(x) = x, \text{ and } \mathcal{R}_y(x) \neq \mathcal{L}_y(y) = y.$$

It follows then that the equality

$$r(xy) = y'x' = \mathcal{L}_x(y)\mathcal{R}_y(x)$$

implies

$$y' \neq x, x' \neq y, \tag{2.1}$$

therefore condition (c) holds. Clearly, (2.1) implies $r(xy) \neq xy$, so the relation $xy = y'x'$ belongs to $\Re_0$. It follows then that every monomial $xy \in X^2 \setminus \Delta_2$ occurs exactly once in $\Re_0$, therefore in $\Re$, which verifies (a) and (b). By [14], Theorem 3.7, the non-degeneracy of r, is equivalent to left and right Ore conditions (see 2.4) on the associated semigroup $\mathcal{S}_0$.

We recall some results which will be used in the paper. The following fact can be extracted from [25].

Fact 2.5. *Suppose A is a standard finitely presented algebra with quadratic Gröbner basis. Then A is Koszul.*

Theorem 2.6. *([14], Theorem 9.7). Let $A = \boldsymbol{k}\langle X\rangle/(\Re)$ be a binomial skew-polynomial ring. Then the automorphism $R = R(\Re) : V^2 \longrightarrow V^2$, associated with $\Re$, is a solution of the Yang-Baxter equation, and (X, r) is (a square-free) set-theoretic solution of the Yang-Baxter equation.*

Conversely, suppose $R : V^2 \longrightarrow V^2$ is a binomial solution of the classical Yang-Baxter equation. Let $\Re = \Re(R) \subset \boldsymbol{k}\langle X\rangle$ be the quadratic binomial relations defined via R. Then X can be enumerated so, that the Yang-Baxter algebra $A = \boldsymbol{k}\langle X\rangle/(\Re)$ is a binomial skew polynomial ring. Furthermore every ordering $\prec$ on X, $X = \{y_1, \cdots, y_n\}$, which makes the relations $\Re$ to be of skew polynomial type, see Definition 1.7 assures that $\Re$ is a Gröbner basis with respect to $\prec$, and the set of ordered monomials $\mathcal{N}_\prec = \{y_1^{\alpha_1} \cdots y_n^{\alpha_n} \mid \alpha_i \geq 0, 1 \leq i \leq n\}$ is a $\boldsymbol{k}$-basis for A.

For the following definition we do not assume Conventions 2.1 necessarily hold.

Definition 2.7. Let $\Xi = \{\xi_1, \cdots \xi_n\}$, be a set of n elements, which is disjoint with X. Let $T^\xi : \langle X\rangle \longrightarrow \langle\Xi\rangle$, be the semigroup isomorphism, extending the assignment $x_i \mapsto \xi_i, 1 \leq i \leq n$. If $\omega = \omega(x) = x_{i_1} \cdots x_{i_k} \in \langle X\rangle$, we call the monomial $T^\xi(\omega) = \xi_{i_1} \cdots \xi_{i_k} \in \langle\Xi\rangle$ *ξ-translation of* ω, and denote it by $\omega(\xi)$. We define the *ξ-translation* of elements $f \in \mathbf{k}\langle X\rangle$, and of subsets $\Re \subset \mathbf{k}\langle X\rangle$ analogously, and use notation $f(\xi)$ and $\Re(\xi)$, respectively. If $\Re_0 = \{\omega_i = \omega_i'\}_{i\in I}$ is a set of semigroup relations in $\langle X\rangle$ by $\Re_0(\Xi)$ we denote the corresponding relations $\Re_0(\Xi) := \{\omega_i(\xi) = \omega_i'(\xi)\}_{i\in I}$ in $\langle\Xi\rangle$.

Clearly the corresponding semigroups are isomorphic:

$$\mathcal{S}_0 = \langle X; \Re_0\rangle \simeq \langle\Xi; \Re_0(\Xi)\rangle$$

and we shall often identify them. Let

$$\mathcal{S}^! = \langle X; \Re_0 \bigcup \{(x_1)^2 = 0, \cdots, (x_n)^2 = 0\}\rangle$$

Then the semigroup $\mathcal{S}^!(\xi)$, associated with $A^!$, see Definition 2.10 is isomorphic to $\mathcal{S}^!$.

Definition 2.8. Let $\mathcal{W} = W(\xi) \in A^!$ be the monomial which spans *the socle*, $A^!_n$ of $A^!$. Then the corresponding monomial $W \in \mathcal{S}_0$, is called *the principle monomial* of A, we shall also refer to it as *the principle monomial* of $\mathcal{S}_0$. A monomial $\omega \in \langle X \rangle$, is called *a presentation of* W if $W = \omega$, as elements of $\mathcal{S}_0$.

Remark 2.9. Clearly, $\mid W(\xi) \mid = n$, so $W(\xi) = \xi_{i_1}\xi_{i_2}\cdots\xi_{i_n}$, for some $i_j, 1 \leq i_j \leq n, 1 \leq j \leq n$. Then the principal monomial $W = x_{i_1}x_{i_2}\cdots x_{i_n} \in \langle X \rangle$, can be considered as a monomial in A, and in the semigroup $\mathcal{S}_0$. Its equivalence class (mod $\Re_0$) in $\langle X \rangle$ contains all monomials $\omega \in \langle X \rangle$, which satisfy $\omega = W$, in $\mathcal{S}_0$. Clearly each such a monomial ω has length n, and is square-free. Furthermore, $\omega = W$, in $\mathcal{S}_0$, if and only if $\omega(\xi) = cW(\xi)$ in $A^!$, for an appropriate $c \in \mathbf{k}^{\times}$.

We will define a special property of W, called *regularity* and will show that it is related to Artin-Schelter regularity of A. More precisely, for a quantum binomial algebra A in which the weak cyclic condition holds, the regularity of the principal monomial W implies Arin-Schelter regularity of A and an analogue of the Poincaré-Birkhoff-Witt theorem for A.

Till the end of the paper we shall often consider (at least) two types of equalities for monomials: a) $u = v$ as elements of $\mathcal{S}_0$ (or in $\mathcal{S}^!$) , and b) $u = v$, as elements of the free semigroup $\langle X \rangle$. We remind that the equality a) means that using the relations $\Re_0$ (or the relations of $\mathcal{S}^!$, respectively) in finitely many steps one can transform u into v (and vice versa). The equality b) means that u and v are equal as words (strings) in the alphabet $\{x_1, \cdots, x_n\}$. Clearly, b) implies a). To avoid ambiguity, when necessary, we shall remind which kind of equality we consider. It follows from the Frobenius property of $A^!$ that every x_i , $1 \leq i \leq n$ occurs as a head (respectively, as a tail) of some presentation of W.

The presentation of the Koszul dual $A^!$, in terms of generators and relations is given in Definition 1.16.

Definition 2.10. If we set $c_{xy} = 1$ for all coefficients in the defining relations of $A^!$, we obtain a new set of relations which define a semigroup with zero. This way we associate naturally to $A^!$, a semigroup with zero denoted by $\mathcal{S}(\xi)^!$. As a set $\mathcal{S}(\xi)^!$ is identified with the set $\mathcal{N} = Nor_{A^!}$ of normal monomials modulo the (uniquely determined) reduced Gröbner basis of $(\Re^{\perp})$. Using the theory of Gröbner basis it is easy to see that for arbitrary $u, v \in \mathcal{N}$ either

a) $uv = 0$ in $A^!$, or

b) $uv = cw$ in $A^!$, with $c \in \mathbf{k}^{\times}$, and $w \in \mathcal{N}$, where the coefficient c and the normal monomial w are uniquely determined, in addition $w \preceq uv$ in $\langle X \rangle$.

We shall often identify $\mathcal{S}(\xi)^!$ with the semigroup $(\mathcal{N}, *)$, where the operation $*$ is defined as follows: for $u, v \in \mathcal{N}$, $u * v := 0$ in case a) and $u * v := w$ in case b).

Remark 2.11. Note that $u * v = 0$ in $\mathcal{S}^!(\xi)$, if and only if the monomial $u(x)v(x)$, considered as a monomial in $\mathcal{S}_0$, has some presentation, which contains a subword of the type xx, where $x \in X$. The shape of the defining relations of A, and the assumption that $A^!$ is Frobenius, imply that a monomial $w \in \langle X \rangle$ is a subword of some presentation of W, $(\omega = W)$ if and only if $w \neq 0$ as an element of $\mathcal{S}^!$.

Definition 2.12. Let $w \in \mathcal{S}_0$. We say that $h \in X$ is *a head of* w if w can be presented (in $\mathcal{S}_0$) as
$$w = hw_1,$$
where $w_1 \in \langle X \rangle$ is a monomial of length $\mid w_1 \mid = \mid w \mid -1$. Analogously, $t \in X$ is *a tail of* w if
$$w = w't \quad (\text{in } \mathcal{S}_0)$$
for some $w' \in \langle X \rangle$, with $\mid w' \mid = \mid w \mid -1$.

It follows from Remark 2.2 b) that for every $i, 1 \leq i \leq n$, there exists a monomial $\omega_i(\xi) \in \langle \Xi \rangle$, such that $\xi_i * \omega_i(\xi) = \mathcal{W}$. Therefore for every $i, 1 \leq i \leq n$, there exists a presentation $W = x_i\omega_i$, with x_i as a head. Similarly, x_i is a tail of W for every $i, 1 \leq i \leq n$. It is not difficult to prove the following.

Lemma 2.13. *The principal monomial W of $\mathcal{S}_0$ satisfies the conditions:*

1. *W is a monomial of length n. There exist $n!$ distinct words $\omega_i \in \langle X \rangle$, $1 \leq i \leq n!$, for which the equalities $\omega_i = W$ hold in $\mathcal{S}_0$. We call them* presentations *of W.*
2. *Every $x \in X$ occurs as a "head" (respectively, as a "tail") of some presentation of W.*
$$W = x_1w_1' = x_2w_2' = \cdots x_nw_n'$$
$$W = \omega_1x_1 = \omega_2x_2 = \cdots \omega_nx_n.$$
3. *No presentation $\omega = W$, where $\omega \in \langle X \rangle$ contains a subword of the form xx, where $x \in X$.*
4. *$W(\xi)$ spans the socle of the Koszul dual algebra $A^!$.*
5. *Every subword a of length k of arbitrary presentation of W, has exactly k distinct "heads", $h_1, \cdots, h_k$, and exactly k distinct "tails" $t_1, \cdots, t_k$.*
6. *W is the shortest monomial which "encodes" all the information about the relations $\Re_0$, More precisely, for any relation $(xy = y'x') \in \Re_0$, there exists an $a \in \langle X \rangle$, such that $W_1 = xya$ and $W_2 = y'x'a$ are (different) presentations of W.*
7. *If $W = ab$ is an equality in $\mathcal{S}_0$, where $a, b \in \langle X \rangle$, then there exists a monomial $b' \in \langle X \rangle$, such that $W = b'a$ in $\mathcal{S}_0$.*

Assume now that there exist a presentation
$$W = y_1y_2 \cdots y_n, \tag{2.2}$$
of W, in which all $y_1, y_2, \cdots, y_n$ are pairwise distinct, that is $y_1, y_2, \cdots, y_n$ is a permutation of $x_1, \cdots, x_n$. (The identity permutation is also allowed). We fix the degree-lexicographic order "$\prec$" on the free semigroup $\langle y_1, \cdots, y_n \rangle = \langle X \rangle$, assuming
$$y_1 \prec y_2 \prec \cdots \prec y_n. \tag{2.3}$$
We say that the order $\prec$ on $\langle X \rangle$ is *associated with the presentation* 2.2.

The theory of Gröbner bases, or the Diamond Lemma, see [5], implies that the set of relations $\Re_0$ determines a unique *reduced Gröbner basis* $\Gamma = \Gamma(\Re_0, \prec)$ in $\langle X \rangle$. In general, Γ is not necessarily finite. In fact, $\Re_0 \subseteq \Gamma$, and every element of Γ is of the form $w = u$, where the monomials $u, w \in \langle X \rangle$ have equal lengths $k \geq 2$, and $u \prec w$. The monomial w is called the *leading monomial* of the relation $w = u$. (Note that the relation $w = u$ follows from $\Re_0$, and holds in $\mathcal{S}_0$.) A monomial $u \in \langle X \rangle$ is called *normal (mod* Γ*)*, if it does not contain as a subword any leading monomial of some element of Γ. Clearly, if u is normal, then any subword u' of

u is normal as well. An important property of the Gröbner basis Γ is that every monomial $w \in \langle X \rangle$ can be reduced (by means of reductions defined by Γ) to a uniquely determined monomial $w_0 \in \langle X \rangle$, which is normal mod Γ, and such that $w = w_0$ is an equality in $\mathcal{S}_0$. In addition $w_0 \preceq w$ always holds in $\langle X \rangle$. The monomial w_0 is called *the normal form* of w and denoted by $Nor_\Gamma(w)$, or shortly $Nor(w)$.

Let $N = N(\Gamma)$ be the set of all normal (mod Γ) monomials in $\langle X \rangle$. As a set $\mathcal{S}_0$ can be identified with N. An operation "$*$" on N is naturally defined as $u * v = Nor(uv)$, which makes $(N, *)$ a semigroup, isomorphic to $\mathcal{S}_0$.

It follows from the definition that there is an equality $\Re_0 = \Gamma$ if and only if $\mathcal{S}_0$ is a semigroup of skew-polynomial type (with respect to the ordering 2.3). The Diamond lemma, [5], provides a recognizable necessary and sufficient condition for $\Re_0$ to be a Gröbner basis: $\Re_0$ is a Gröbner basis with respect to $\prec$, if and only if every monomial of the shape $y_k y_j y_i$, with $n \geq k > j > i \geq 1$, can be reduced using $\Re_0$ to a uniquely determined monomial of the shape $y_p y_q y_r$, with $p \leq q \leq r$.

Definition 2.14. Let $W = W(r)$ be the principal monomial of $\mathcal{S}_0$. We say that $W = y_1 y_2 \cdots y_n$, is a *regular presentation* of W if the following two conditions are satisfied:

(1) $y_1, y_2, \cdots, y_n$ is a permutation of $x_1, \cdots, x_n$; and
(2) $y_1 y_2 \cdots y_n$ is the minimal presentation of W with respect to $\prec$ in $\langle X \rangle$ (i.e. each $\omega \in \langle X \rangle$, such that $\omega = W$ in S, satisfies $y_1 y_2 \cdots y_n \prec \omega$).
In this case we also say that $\prec$ is a *regular order* in $\langle X \rangle$

We say that the Koszul dual $A^!$ has *a regular socle*, if the principal monomial W has a regular presentation.

Remark 2.15. Let $W = y_1 y_2 \cdots y_n$ be a regular presentation of W. It follows from the definition 2.14 that $Nor(W) = y_1 y_2 \cdots y_n$, or equivalently, the monomial $y_1 y_2 \cdots y_n$ is normal (mod Γ.) Clearly, every subword of $y_1 y_2 \cdots y_n$ is normal as well. In particular, the monomial $y_j y_{j+1}$ is normal for every $j, 1 \leq j < n$. Thus $y_j y_{j+1} = zt \in \Re$ implies $z \succ y_j, t \neq y_{j+1}$.

Example 2.16. Let $X = \{x_1, x_2, x_3, x_4\}$, $S = \langle X; \Re_0 \rangle$ be the semigroup with defining relations $\Re_0$:

$$\begin{array}{lll} x_1x_2 = x_3x_4 & x_1x_3 = x_2x_4 & x_4x_2 = x_3x_1 \\ x_4x_3 = x_2x_1 & x_1x_4 = x_4x_1 & x_2x_3 = x_3x_2. \end{array}$$

Then the relations $\Re$ define a set-theoretic solution (X, r) of the Yang-Baxter equation, therefore by [14], $A^!$ is Frobenius. Furthermore $x_1x_2x_3x_4 = W$ is a presentation of W as a product of pairwise distinct elements of X, but this presentation is not regular. In fact, the monomial x_3x_4 is a sub monomial of W, but it is not normal, since $x_3x_4 = x_1x_2$ (in S) and $x_1x_2 \prec x_3x_4$. Nevertheless W has regular presentations. For example each of the monomials in the following equalities gives a regular presentation of $W : x_2x_3x_1x_4 = x_1x_4x_2x_3 = x_4x_1x_3x_2 = W$.

Lemma 2.17. *$\mathcal{S}^!$ has a cancelation law on nonzero products. More precisely, if $a, b, c \in \mathcal{S}^!$ then i) $ab = ac \neq 0$ implies $b = c$; ii) $ba = ca \neq 0$ implies $b = c$.*

Proof. Conditions i) and ii) are analogous. We shall prove i) using induction on the length m of a.

Step 1. Let $\mid a \mid = 1$, so $a \in X$. Suppose for some monomials b and c one has:

$$ab = ac \neq 0.$$

It follows then that ab, ac, and therefore b and c are subwords of W. Clearly b and c have equal lengths,

$$\mid b \mid=\mid c \mid= k,\ \ k \geq 1.$$

In the case when $k = 1$, the equality $ab = ac$ can not be a relation because of the nondegeneracy property, therefore it is simply equality of words in the free semigroup $\langle X \rangle$, so $b = c \in X$. Assume now that the length $k \geq 2$, and,

$$b \neq c. \tag{2.4}$$

Note that each of the monomials b and c has exactly k heads, as a subword of W. Let $H_b = \{b_1, \cdots, b_k\}$ be the set of all heads of b and $H_c = \{c_1, \cdots, c_k\}$ be the set of heads of c. The inequality 2.4 implies that

$$H_b \neq H_c. \tag{2.5}$$

The following relations hold in $\mathcal{S}_0$, for appropriate $b'_i, c'_i, a_i, a^{(i)} \in X$, $1 \leq i \leq k$.

$$\begin{array}{c} ab_i = b'_i a_i, 1 \leq i \leq k, \\ ac_i = c'_i a^{(i)}, 1 \leq i \leq k. \end{array} \tag{2.6}$$

It follows from (2.5) and the non-degeneracy property that there is an inequality of sets:

$$\{b'_i \mid 1 \leq i \leq k\} \neq \{c'_i \mid 1 \leq i \leq k\}.$$

Clearly, then the set of heads of the monomial $ab = ac$ is

$$H_{ab} = \{a\} \bigcup \{b'_i \mid 1 \leq i \leq k\} \bigcup \{c'_i \mid 1 \leq i \leq k\}.$$

By the nondegeneracy condition one has $a \neq b'_i$, $a \neq c'_i$, which together with (**??**) imply that H_{ab} contains at least $k + 2$ elements. This gives a contradiction, since the monomial ab is a subword of W therefore the number of its heads equals its length $\mid ab \mid= k + 1$.

Step 2. Assume the statement of the lemma is true for all monomials a, b, c, with $\mid a \mid\leq m$. Suppose $ab = ac \neq 0$ holds in $\mathcal{S}^!$, where $\mid a \mid= m + 1$. Let $a = z_1 \cdots z_{m+1}$. Therefore $z_1 * (z_2 \cdots z_{m+1} * b) = z_1 * (z_2 \cdots z_{m+1} * c)$, which by the inductive assumption implies first that $(z_2 \cdots z_{m+1} * b) = (z_2 \cdots z_{m+1} * c)$, and again by the inductive assumption one has $b = c$.

□

Remark 2.18. In some cases, when we study quadratic algebras, instead of applying reductions to monomials of length 3 (in the sense of Bergman [5]), it is more convenient to study the action of the infinite dihedral group, $\mathcal{D}(\Re)$ generated by maps associated with the quadratic relations, as it is suggested below.

Let $\Re$ be quantum binomial relations, $r = r(\Re)$ the associated bijective map $r : X^2 \longrightarrow X^2$. Clearly the two bijective maps $r_{ii+1} : X^3 \longrightarrow X^3$, $1 \leq i \leq 2$, where $r_{12} = r \times Id_X$, and $r_{23} = Id_X \times r$ are involutive. The infinite dihedral group,

$$\mathcal{D} = \mathcal{D}(r) = \ _{\mathrm{gr}}\langle r_{12}, r_{23} : r_{12}^2 = e, r_{23}^2 = e\rangle$$

acts on X^3. The orbit $\mathcal{O}_{\mathcal{D}}(\omega)$ of $\omega \in X^3$ consists of all monomials $\omega' \in X^3$ such that $\omega' = \omega$ is an equality in $\mathcal{S}_0$. Clearly each reduction ρ applied to a monomial $\upsilon \in X^3$ can be presented as $\rho(\upsilon) = r^{ii+1}(\upsilon)$, where $1 \leq i \leq 2$. So every monomial ω' obtained by a sequence of reductions applied to ω belongs to $\mathcal{O}_{\mathcal{D}}(\omega)$. The convenience of this approach is that it does not depend on the enumeration of X (therefore on the chosen order $\prec$ on $\langle X \rangle$.

Lemma 2.19. *Suppose the quantum binomial algebra $A = k\langle X;\Re\rangle$ satisfies the weak cyclic condition, 1.14. Let $\mathcal{O} = \mathcal{O}_{\mathcal{D}}(\omega)$ be an arbitrary orbit of the action of $\mathcal{D}$ on X^3. Then the following conditions hold.*

1. *$\mathcal{O}\bigcap\Delta_3 \neq \emptyset$ if and only if $\mathcal{O} = \{xxx\}$, for some $x \in X$.*
2. *$\mathcal{O}\bigcap((\Delta_2 \times X\bigcup X \times \Delta_2)\backslash\Delta_3)) \neq \emptyset$ if and only if $\mid\mathcal{O}\mid= 3$.*
3. *In each of the cases $\omega = yyx$, or $\omega = yxx$, where $x, y \in X, x \neq y$, the orbit $\mathcal{O}_{\mathcal{D}}(\omega)$ contains exactly 3 elements. More precisely, if (by the weak cyclic condition) the following are relations in $\mathcal{S}_0$:*

$$yx = x_1y_1, \;\; yx_1 = x_2y_1 \;\text{ and }\; y_1x_1 = x_2y_2,$$

then there are equalities of sets:

$$\mathcal{O}_{\mathcal{D}}(yyx) = \{yyx, yx_1y_1, x_2y_1y_1\}$$
$$\mathcal{O}_{\mathcal{D}}(yxx) = \{yxx, x_1y_1x, x_1x_1y_2\}.$$

Furthermore, suppose $\prec$ is an ordering on X such that every relation in $\Re_0$ is of the type $yx = x'y'$, where $y \succ x$, $x' \prec y'$, and $y \succ x'$. Then the orbit $\mathcal{O}_{\mathcal{D}}(y_1y_2y_3)$ with $y_1 \prec y_2 \prec y_3$ does not contain elements of the form xxy, or xyy, $x \neq y \in X$.

Theorem 2.20. *Let $A = A(\boldsymbol{k}, X, \Re) = k\langle X;\Re\rangle$ be a quantum binomial algebra, let $\mathcal{S}_0 = \langle X;\Re_0\rangle$ be the associated semigroup, and let $A^!$ be the Koszul dual of A. We assume that the following conditions are satisfied:*

(a) *The weak cyclic condition is satisfied on $\mathcal{S}_0$.*
(b) *The Koszul dual $A^!$ is Frobenius.*
(c) *The principal monomial W has a regular presentation $W = y_1y_2\cdots y_n$.*

Then $\mathcal{S}_0 = \langle y_1, y_2,\cdots, y_n;\Re_0\rangle$ is a semigroup of skew-polynomial type (with respect to the order $\prec$, where $y_1 \prec y_2 \prec\cdots\prec y_n$). More precisely, the following conditions hold:

1. *Each relation in $\Re_0$, is of the form $yz = z'y'$, where $y \succ z$ implies $z' \prec y'$, and $y \succ z'$.*
2. *The relations $\Re_0$ form a Gröbner basis with respect to the ordering $\prec$ on $\langle X\rangle$.*
3. *The relations $\Re$ form a Gröbner basis with respect to the (degree-lexicographic) ordering $\prec$ on $\langle X\rangle$, and A is a binomial skew-polynomial ring.*
4. *The set of ordered monomials*

$$\mathcal{N} = \{y_1^{\alpha_1}\cdots y_n^{\alpha_n} \mid \alpha_i \geq 0, 1leqi \leq n\}$$

forms a $\boldsymbol{k}$-basis of A. That is A is a PBW-type algebra.

5. *A is Koszul.*
6. *A is Artin-Schelter regular ring of global dimension n.*

We assume conditions (a), (b) , (c) of the hypothesis of the theorem are satisfied and prove two more statements.

Proposition 2.21. *The following conditions hold in $\mathcal{S}_0$.*

1. *For any integer j, $1 \leq j \leq n-1$, there exists a unique $\eta_j \in X$, such that*

$$y_{j+1}\cdots y_n\eta_j = y_jy_{j+1}\cdots y_n.$$

2. *The elements $\eta_1, \eta_2,\cdots, \eta_{n-1}$ are pairwise distinct.*

(3) *For each* j, $1 \leq j \leq n-1$, *the set of heads* H_{W_j} *of the monomial* $W_j = y_j y_{j+1} \cdots y_n$ *is*
$$H_{W_j} = \{y_j, y_{j+1}, \cdots, y_n\}.$$
(4) *For any pair of integers* i, j, $1 \leq i < j \leq n$, *the monomial* $y_i y_j$ *is normal. Furthermore, the unique relation in which* $y_i y_j$ *occurs has the form* $y_{j'} y_{i'} = y_i y_j$, *with* $j' > i'$, *and* $j' > i$.

Lemma 2.22. *For each integer* j, $1 \leq j \leq n-1$, *let* $\xi_{j,j+1}, \cdots, \xi_{j,n}$, $\eta_{j,j+1}, \cdots, \eta_{j,n}$ *be the elements of* X *uniquelly determined by the relations*

$$\begin{aligned}
&\xi_{j,j+1}\eta_{j,j+1} = y_j y_{j+1} \in \Re_0 \\
&\xi_{j,j+2}\eta_{j,j+2} = \eta_{j,j+1} y_{j+2} \in \Re_0 \\
&\cdots\cdots\cdots \\
&\xi_{j,n-1}\eta_{j,n-1} = \eta_{j,n-2} y_{n-1} \in \Re_0 \\
&\xi_{j,n}\eta_{j,n} = \eta_{j,n-1} y_n \in \Re_0
\end{aligned} \tag{2.7}$$

Then for each j, $1 \leq j \leq n-1$, *the following conditions hold:*

(1) $\xi_{j,j+s} \neq \eta_{j,j+s-1}$, *for all* $s, 2 \leq s \leq n-j$.
(2) *There is an equality of in* S*:*
$$\xi_{j,j+1}\xi_{j,j+2} \cdots \xi_{j,n} = y_{j+1} \cdots y_n.$$
(3) $y_{j+1} y_{j+2} \cdots y_n \eta_{j,n} = y_j y_{j+1} \cdots y_n$.
(4) *The elements* $\eta_{j,n}, \eta_{j+1,n}, \cdots, \eta_{n-1,n}$ *are pairwise distinct.*

Proof. Condition (1) is obvious. To prove the remaining conditions we use decreasing induction on j, $1 \leq j \leq n-1$.

Step 1. $j = n-1$. Clearly, $y_{n-1}y_n$ is normal thus (see Remark 2.15) the relation in $\Re_0$ in which it occurs has the shape $y_{n-1}y_n = \xi_{n-1,n}\eta_{n-1,n}$, with $\xi_{n-1,n} \succ y_{n-1}$. It follows then that $\xi_{n-1,n} = y_n$ and $y_{n-1}y_n = y_n\eta_{n-1,n},$, so $y_{n-1}y_n = y_{n-1} \vee y_n.$. This gives (2). (3) and (4) are clear.

Step 2. We first prove (4) for all $j, 1 \leq \jmath \leq n-1$. Assume that for all k, $n-1 \geq k > j$, the elements $y_k, y_{k+1}, \cdots, y_n$, $\xi_{k,k+1}, \cdots, \xi_{k,n}$, $\eta_{k,k+1}, \cdots, \eta_{k,n}$ satisfy

$$\begin{aligned}
&\xi_{k,k+1}\eta_{k,k+1} = y_k y_{k+1} \in \Re_0 \\
&\xi_{k,k+2}\eta_{k,k+2} = \eta_{k,k+1} y_{k+2} \in \Re_0 \\
&\cdots\cdots\cdots \\
&\xi_{k,n-1}\eta_{k,n-1} = \eta_{k,n-2} y_{n-1} \in \Re_0 \\
&\xi_{k,n}\eta_{k,n} = \eta_{k,n-1} y_n \in \Re_0,
\end{aligned} \tag{2.8}$$

all $\eta_{j+1,n}, \eta_{j+2,n}, \cdots, \eta_{n-1,n}$ are pairwise distinct, and the modified conditions (4), in which "j" is replaced by "k" hold. Let $\xi_{j,j+1}, \cdots \xi_{j,n}$, $\eta_{j,j+1} \cdots \eta_{j,n}$ satisfy (2.8). We shall prove that $\eta_{j,n} \neq \eta_{k,n}$, for all $k, j < k \neq n-1$. Assume the contrary,
$$\eta_{j,n} = \eta_{k,n}$$
for some $k > j$. Consider the relations
$$\xi_{j,n}\eta_{j,n} = \eta_{j,n-1}y_n, \quad \xi_{k,n}\eta_{k,n} = \eta_{k,n-1}y_n. \tag{2.9}$$
The Ore condition, (see Definition 2.4), and (2.9) imply
$$\eta_{j,n-1} = \eta_{k,n-1}.$$

Using the same argument in $n-k$ steps we obtain the equalities

$$\eta_{j,n} = \eta_{k,n}, \eta_{j,n-1} = \eta_{k,n-1}, \cdots, \eta_{j,k+1} = \eta_{k,k+1}.$$

Now the relations

$$\xi_{j,k+1}\eta_{j,k+1} = \eta_{j,k}y_{k+1},\ \xi_{k,k+1}\eta_{k,k+1} = y_k y_{k+1},$$

and the Ore condition imply $\eta_{j,k} = y_k$. Thus, by (2.8) and (2.7) we obtain a relation

$$\xi_{j,k}y_k = \xi_{j,k-1}y_k \in \Re_0.$$

This is impossible, by Lemma 1.5 (iii). We have shown that the assumption $\eta_{j,n} = \eta_{k,n}$, for some $k > j$, leads to a contradiction. This proves (4) for all $j, 1 \le j \le n-1$.

We set

$$\eta_1 = \eta_{1,n}, \eta_2 = \eta_{2,n}, \cdots, \eta_{n-1} = \eta_{n-1,n}. \tag{2.10}$$

Next we prove (2) and (3).

By the inductive assumption, for $k > j$, we have

$$\xi_{k,k+1}\xi_{k,k+2}\cdots\xi_{k,n} = y_{k+1}\cdots y_n,$$

and

$$y_{k+1}\cdots y_n.\eta_{k+1} = y_k\cdots y_n.$$

Applying the relations (2.8) one easily sees, that

$$\xi_{j,j+1}\xi_{j,j+2}\cdots\xi_{j,n}.\eta_{j,n} = y_j y_{j+1}\cdots y_n.$$

Denote

$$\omega_j = \xi_{j,j+1}\xi_{j,j+2}\cdots\xi_{j,n}.$$

We have to show that the normal form, $Nor(\omega_j)$, of ω_j satisfies the equality of words

$$Nor(\omega_j) = y_{j+1}y_{j+2}\cdots y_n, \quad \text{in } \langle X\rangle.$$

As a subword of length $n-j$ of the presentation $W = y_1y_2\cdots y_{j-1}w_j\eta_{j,n}$, the monomial ω_j has exactly $n-j$ heads

$$h_1 \prec h_2 \prec \cdots \prec h_{n-j}. \tag{2.11}$$

Since $Nor(\omega_j) = \omega_j$, is an equality in $\mathcal{S}_0$, the monomial $Nor(\omega_j)$ has the same heads as ω_j. Furthermore, there is an equality of words in $\langle X\rangle$, $Nor(\omega_j) = h_1\omega'$, where ω' is a monomial of length $n-j-1$. First we see that $h_1 \succeq y_j$ This follows immediately from the properties of the normal monomials and the relations

$$Nor(\omega_j)\eta_j = \omega_j\eta_j = y_jy_{j+1}\cdots y_n \in N. \tag{2.12}$$

Next we claim that $h_1 \succ y_j$. Assume the contrary, $h_1 = y_j$. Then by (2.12) one has

$$y_j\omega'\eta_j = \omega_j\eta_j = y_jy_{j+1}\cdots y_n.$$

The cancellation law in $\mathcal{S}_0$ implies that

$$\omega'\eta_j = y_{j+1}\cdots y_n \in N.$$

Thus η_j is a tail of the monomial $y_{j+1}\cdots y_n$. By the inductive assumption, conditions (2) and (3) are satisfied, which together with (2.10) give additional $n-j$ distinct tails of the monomial $y_{j+1}\cdots y_n$, namely $\eta_{j+1}, \eta_{j+2}, \cdots \eta_{n-1}, y_n$. It follows then that the monomial $y_{j+1}\cdots y_n$ of length $n-j$ has $n-j+1$ distinct tails, which

is impossible. This implies $h_1 \succ y_j$. Now since ω_j has precisely $n-j+1$ distinct heads, which in addition satisfy (2.11) we obtain equality of sets

$$\{h_1, h_2, \cdots, h_{n-j}\} = \{y_{j+1}, y_{j+2}, \cdots, y_n\}.$$

By the inductive assumption the heads of the monomial $y_{j+1}y_{j+2}\cdots y_n$ are exactly $y_{j+1}, y_{j+2}, \cdots, y_n$, therefore, there is an equality

$$\omega_j = y_{j+1}y_{j+2}\cdots y_n \quad \text{in } \mathcal{S}_0.$$

We have shown (3). Now the equality

$$y_{j+1}\cdots y_n\eta_j = y_j y_{j+1}\cdots y_n$$

and the inductive assumption give that the heads of $y_j y_{j+1}\cdots y_n$ are precisely the elements $y_j, y_{j+1}, \cdots, y_n$. This proves (2). The lemma has been proved. $\square$

Proof of Proposition 2.21. Conditions (1), (2), (3) of the proposition follow from Lemma 2.22. We shall prove first that for any pair i, j, $1 \leq i < j \leq n$, the monomial $y_i y_j$ is normal. Assume the contrary. Then there is a relation

$$(y_i y_j = y_{j'} y_{i'}) \in \Re_0, \tag{2.13}$$

where

$$y_{j'} \prec y_i.$$

Consider the monomial

$$u = y_i y_j . y_{j+1}\cdots y_n \eta_{j-1}\eta_{j-2}\cdots\eta_{i+1}. \tag{2.14}$$

We replace 2.13 in 2.14 and obtain

$$u = y_{j'} y_{i'} y_{j+1}\cdots y_n \eta_{j-1}\eta_{j-2}\cdots\eta_{i+1},$$

so $y_{j'}$ is one of the heads of u. It follows from 2.22.3 that there is an equality in $\mathcal{S}_0$ $u = y_i y_{i+1}\cdots y_n = Nor(u)$. Since the inequality $Nor(u) \preceq u$ always holds in $\langle X\rangle$, y_i is the smallest "head" of u. But, by our assumption, the head $y_{j'}$ of u satisfies $y_{j'} \prec y_i$, which gives a contradiction. We have proved that the monomial $y_i y_j$ is normal for every pair i, j, $1 \leq i < j \leq n$. Since the number of relations is exactly $\binom{n}{2}$ and each relation contains exactly one normal monomial, this implies that all monomials $x_j x_i$, with $1 \leq i < j \leq n$, are not normal. It follows then that each relation in $\Re_0$ has the shape $y_j y_i = y_{i'} y_{j'}$, where $1 \leq i < j \leq n$, $1 \leq i' < j' \leq n$, and $j > i'$, which proves (3) and (4).

Lemma 2.23. *The following conditions hold.*

(a) *The set of relations $\Re_0$ is Gröbner basis with respect to the ordering $\prec$ on $\langle X\rangle$.*
(b) *$\mathcal{S}_0$ is a semigroup of skew polynomial type.*
(c) *(X, r) is a square-free solution of the set-theoretic Yang-Baxter equation.*
(d) *$\Re$ is a Gröbner basis of the ideal $(\Re)$*
(e) *A is a binomial skew polynomial ring.*
(f) *The automorphism $R = R(\Re)$ is a solution of the classical Yang-Baxter equation;*

Proof. We denote by Γ the reduced Gröbner basis of the ideal $(\Re_0)$ and claim that $\Gamma = \Re_0$. It will be enough to prove that the ambiguities $y_k y_j y_i$, with $k > j > i$, do

not give rise to new relations in $\mathcal{S}_0$. Or equivalently, the set $\mathcal{N}_3$ of all monomials of length 3 which are normal modulo $\Re_0$:

$$\mathcal{N}_3 = \{xyz \mid x, y, z \in X, \text{ and } x \preceq y \preceq z\},$$

coincides with $N_3 = N \bigcap X^3$, the set of all monomials of length 3 which are normal modulo Γ. Clearly, $N_3 \subseteq \mathcal{N}_3$.

Let $\omega \in \mathcal{N}_3$. We have to show $Nor_\Gamma(\omega) = \omega$. Four cases are possible:

$$\begin{array}{ll} (i) & \omega = y_i y_j y_k, 1 \leq i < j < k \leq n \\ (ii) & \omega = y_i y_i y_j, 1 \leq i < j \leq n \\ (iii) & \omega = y_i y_j y_j, 1 \leq i < j \leq n \\ (iv) & \omega = y_i y_i y_i, 1 \leq i \leq n. \end{array} \tag{2.15}$$

Case 1. Suppose 2.15 (i) holds. Assume, on the contrary, ω is not in N_3. Then there is an equality

$$\omega = y_i y_j y_k = y_i' y_j' y_k', \text{ where } y_i' \preceq y_j' \preceq y_k',$$

and as elements of $\langle X \rangle$, the two monomials satisfy

$$y_i' y_j' y_k' \prec y_i y_j y_k. \tag{2.16}$$

By (2.16) one has

$$y_i' \preceq y_i. \tag{2.17}$$

We claim that there is an inequality $y_i' \prec y_i$. Indeed, it follows from Lemma 2.19 that the orbit $\mathcal{O}_\mathcal{D}(y_i y_j y_k)$ does not contain elements of the shape xxy, or xyy, therefore an assumption, $y_i = y_i'$ would imply $y_j y_k = y_j' y_k'$ with $y_j \prec y_k$ and $y_j' \prec y_k'$, which contradicts Proposition 2.21. We have obtained that $y_i' \preceq y_i$. One can easily see that there exists an $\omega \in \langle X \rangle$, such that

$$(y_i y_j y_k) * \omega = y_i y_{i+1} \cdots y_n.$$

The monomial $y_i y_{i+1} \cdots y_n$ is normal, therefore the normal form $(y_i y_j y_k) * \omega$ satisfies

$$Nor(y_i' y_j' y_k' * \omega) = Nor(y_i y_j y_k * \omega) = y_i y_{i+1} \cdots y_n.$$

Now the inequalities

$$Nor(y_i' y_j' y_k' * \omega) \preceq y_i' y_j' y_k' \prec y_i y_{i+1} \cdots y_n$$

give a contradiction. It follows then that monomial $y_i y_j y_k$, $i < j < k$, is normal mod Γ.

Case 2. $\omega = y_i y_i y_j, 1 \leq i < j \leq n$. It is not difficult to see that the orbit $\mathcal{O} = \mathcal{O}_\mathcal{D}(y_i y_i y_k)$ is the set

$$\mathcal{O} = \{\omega = y_i y_i y_k, \omega_1 = y_i y_k' y_i', \omega_3 = y_k'' y_i' y_i'\}$$

where

$$y_k' y_i' = y_i y_k \in \Re_0, \text{ with } y_i \prec y_k' \succ y_i'$$

and

$$y_k'' y_i' = y_i y_k' \in \Re_0, \text{ with } y_k'' \succ y_i'.$$

Therefore

$$Nor(y_i y_i y_k) \in \mathcal{O}_\mathcal{D}(y_i y_i y_k) \bigcap \mathcal{N}_3 = y_i y_i y_j$$

We have shown that $Nor_\Gamma(\omega) = \omega$.

Case 3 is analogous to Case 2. **Case 4.** is straightforward, since all relations are square free. We have proved condition (a).

Condition (b) is straightforward.

We have shown that $\mathcal{S}_0 = \langle X; \Re_0 \rangle$ is a semigroup of skew polynomial type. Clearly $r = r(\Re) = r(\Re_0)$. It follows then from [16], Theorem 1.1 that (X, r) is a solution of the set-theoretic Yang-Baxter equation which proves (c).

We shall prove (d). It will be enough to show that each ambiguity $\omega = y_k y_j y_i$, with $k > j > i$ is solvable.

Note first that since (X, r) is a solution of the Yang-Baxter equation, the group $\mathcal{D} =$ is isomorphic to the dihedral group $\mathcal{D}_3$, and each monomial of length 3 has an orbit consisting either of 1, or 3 or 6 elements. Furthermore the orbit $\mathcal{O}_{\mathcal{D}}(\omega)$ consists of exactly 6 elements. This follows directly from lemma 2.19 (this was proven first in [16]). Furthermore $\mathcal{O}_{\mathcal{D}}(\omega)$ contains exactly one ordered monomial $\omega_0 = y_{i_1} y_{j_1} y_{k_1}$, with $1 \leq i_1 < j_1 < k_1 \leq n$, which is the normal form of ω (mod $\Re_0$). Two cases are possible. Either

$$r^{12} r^{23} r^{12}(\omega) = \omega_0 = r^{23} r^{12} r^{23}(\omega)$$

or

$$r^{12} r^{23} r^{12} r^{23}(\omega) = \omega_0 = r^{23} r^{12}(\omega).$$

Denote by $\mathcal{O}_{\Re}(\omega)$ the set of all elements $f \in A$, which can be obtained by finite sequences of reductions, defined via the set of relations $\Re$ (in the sense of [5]) applied to ω. In fact each reduction ρ applied to a monomial of length 3, which is not fixed under ρ behaves as one of the automorphism R^{12} and R^{23} but only in one direction, transforming each monomial ω' which is not ordered into $\rho(\omega') = c_{pq}\omega''$, where c_{pq} is the coefficient occurring in the relation used for ρ and $\omega'' \prec \omega'$. So each $f \in \mathcal{O}_{\Re}(\omega)$ has the shape $f = c\omega^{\star}$, where $c \in \mathbf{k}^{\times}$, and $\omega^{\star}$ is in the orbit $\mathcal{O}_{\mathcal{D}}(\omega)$. We know only that $\mathcal{O}_{\Re}(\omega)$ contains 6 elements, but the normal form, ω_0, might occur with two distinct coefficients.

Assume now that the ambiguity $y_k y_j y_i$ is not solvable. Then the orbit $\mathcal{O}_{\Re}(\omega)$ contains two distinct elements $c_1\omega_0$ and $c_2\omega_0$, with $c_1, c_2 \in \mathbf{k}^{\times}$, and $c_1, \neq c_2$. On the other hand every $f \in \mathcal{O}_{\Re}(\omega)$ satisfies $f \equiv \omega$ (modulo $(\Re)$). It follows then $\omega_0 \in (\Re)$. One can find appropriate $\eta_{s_1}, \cdots \eta_{s_{n-3}} \in X$ where η_j, $1 \leq j \leq n$, are as in Proposition 2.21 so that the following equality holds in $\mathcal{S}_0$:

$$y_{i_1} y_{j_1} y_{k_1} \eta_{s_1} \cdots \eta_{s_{n-3}} = W.$$

But then there is an equality in A

$$y_{i_1} y_{j_1} y_{k_1} \eta_{s_1} \cdots \eta_{s_{n-3}} = \alpha W$$

for some $\alpha \in \mathbf{k}^{\times}$. The element αW is in normal form therefore, $y_{i_1} y_{j_1} y_{k_1} = 0$ in A leads to a contradiction. It follows then that each ambiguity $y_k y_j y_i$, with $k > j > i$, is solvable. Therefore $\Re$ is a Gröbner basis, and A is a binomial skew polynomial ring. This proves conditions (d) and (e). It follows from [14], Theorem 9.7 (see also Theorem 2.6) that the automorphism $R(\Re)$ is a solution of the classical Yang-Baxter equation. □

Proof of Theorem 2.20. Condition (1) follows from Proposition 2.21. (4). Lemma 2.23 implies (2) and (3). Clearly (3) implies (4). It is known that every standard finitely presented algebra with quadratic Gröbner basis is Koszul, Fact 2.5, which implies (5). We have already shown that A is a binomial skew polynomial ring. It follows then from the proof of Theorem 3.1 that A has global

dimension n. Now a result of Stafford and Zhang, [29] see also P. Smith's, see 3.2, implies that A is Gorenstein and therefore, A is Artin-Schelter regular.

3. The koszul dual of a binomial skew polynomial ring is Frobenius

In this section we study the Koszul dual $A^!$ of a binomial skew polynomial ring A. We prove Theorem A, which guarantees Frobenius property for a class of quadratic algebras with specific relations. This class includes the Koszul dual $A^!$. The main result of the section, 3.1, shows that the binomial skew polynomial rings with n generators provide a class of Artin-Schelter regular rings of global dimension n. The first proof of this theorem (1995) was given in [11], where we used combinatorial methods to show that $A^!$ is Frobenius, and then a result of P. Smith , to show that A is regular. In [16] this result was improved by a different argument, which uses the good algebraic and homological properties of semigroups of I-type to show that A is an Artin-Schelter regular domain. We present here the original combinatorial proof of the Frobenius properties of $A^!$, which has not been published yet and uses a technique which might be useful in other cases of (standard) finitely presented algebras.

Theorem 3.1. *Let A be a skew-polynomial ring with binomial relations. Then*

(1) *The Koszul dual $A^!$ is Frobenius.*
(2) *A is Artin-Schelter regular ring of global dimension n.*

Our proof is combinatorial, we deduce the Frobenius property of $A^!$ from its defining relations. We use Gröbner basis techniques, the cyclic condition in A, and study more precisely the computations in the associated semigroup $\mathcal{S}^!$. Next we recall the following result which will be used to deduce the Gorenstein property of A.

Proposition 3.2. [28]*, Proposition 5.10. Let A be a Koszul algebra of finite global dimension. Then A is Gorenstein if and only if $A^!$ is Frobenius.*

We keep the notation from the previous sections. As before we denote the set of generators of $A^!$ as $\Xi = \{\xi_1, \xi_2, \cdots, \xi_n.\}$

Remark 3.3. In [10], Theorem 3.16 (see also [9]) was shown that every binomial skew polynomial ring A satisfies *the cyclic condition*, a condition stronger than the weak cyclic condition , see Definition 1.14. So the algebra A, satisfies the conditions of Definition 1.14. One can easily deduce from the relations of $A^!$, see Notation 3.4 that it also satisfies the conditions of Definition 1.14.

We need the explicit relations of $A^!$.

Let $A = \mathbf{k}\langle X\rangle/(\Re)$ be a binomial skew-polynomial ring, with a set of relations

$$\Re = \{x_j x_i - c_{ij} x_{i'} x_{j'}\}_{1\leq i<j\leq n}. \tag{3.1}$$

where for each pair $1 \leq i < j \leq n$, the relation $x_j x_i - c_{ij} x_{i'} x_{j'}$, satisfies $j > i'$, $i' < j'$, and $c_{ij} \in \mathbf{k}^\times$. Furthermore, the relations $\Re$ form a Gröbner basis, with respect to the degree-lexicographic order on $\langle X\rangle$.

Notation 3.4. Let $\Xi = \{\xi_1, \cdots, \xi_n\}$ be a set of indeterminates, $\Xi \bigcap X = \emptyset$. Consider the following subsets of the free associative algebra $\mathbf{k}\langle\Xi\rangle$:

$$\Re^* = \{\xi_j \xi_i + (c_{ij})^{-1} \xi_{i'} \xi_{j'}\}_{1\leq i<j\leq n}.$$

We call $\Re^*$ *the dual relations, associated to* $\Re$. Let

$$\begin{array}{lllllll} \Re_1 & = & \{(x_j)^2\}_{1\leq j\leq n}, & & \Re^! & = & \Re\bigcup\Re_1, \\ \Re_1^* & = & \{(\xi_j)^2\}_{1\leq j\leq n}, & & \Re^\perp & = & \Re^*\bigcup\Re_1^*. \end{array}$$

It follows from the definition of Koszul dual 1.16 that:

Remark 3.5. Let $A = \mathbf{k}\langle X\rangle/(\Re)$ be a binomial skew-polynomial ring, with a set of relations $\Re$ as in (3.1). Then the Koszul dual $A^!$ has the following presentation via generators and relations:

$$A^! = \mathbf{k}\langle\Xi\rangle/(\Re^\perp). \tag{3.2}$$

The next lemma is straightforward

Lemma 3.6. *Let $\omega \in \langle X\rangle$. Suppose $\Re \subset \boldsymbol{k}\langle X\rangle$ is a set of quantum binomial relations, and $\Re^* \subset \boldsymbol{k}\langle\Xi\rangle$, is the associated dual relation set. Let $\Re_0$ and $\Re_0^*$, respectively, be the semigroup relations associated with $\Re$ and $\Re^*, see Definition 1.4.$ Then the following conditions hold:*

1. *There is an equality $(\omega(\xi))(x) = \omega$.*
2. $(\Re_0)(\xi) = (\Re^*)_0 = (\Re_0)^*$.
3. *The ξ-translation isomorphism T^ξ induces (semigroup) isomorphisms:*
   a) between the associated semigroups
   $$\mathcal{S}_0 = \mathcal{S}_0(X,\Re_0) = \langle X;\Re_0\rangle \simeq \mathcal{S}_0(\xi) = \mathcal{S}_0(\Xi,\Re_0^*) = \langle\Xi;\Re_0^*,\rangle$$
   and
   b) between the "Koszul-type" semigroups
   $$\mathcal{S}^! = \langle X;\Re_0\bigcup\Re_1\rangle \simeq (\mathcal{S}(\xi))^! = \langle\Xi;\Re_0^*\bigcup\Re_1^*\rangle.$$

For our purposes it will be often more convenient to perform computations and arguments in $\mathcal{S}_0$, $\mathcal{S}^!$ and A, respectively, and then "translate" the results for $\mathcal{S}_0(\xi)$, $(\mathcal{S}(\xi))^!$ and $A^!$.

Lemma 3.7. *In notation 3.4, the following conditions are equivallent:*

1. *$\Re$ is a Gröbner basis of the ideal $(\Re)$ in $\boldsymbol{k}\langle X\rangle$.*
2. *$\Re^*$ is a Gröbner basis of the ideal $(\Re^*)$ in $\boldsymbol{k}\langle\Xi\rangle$.*
3. *$\Re^!$ is a Gröbner basis of the ideal $(\Re^!)$ in $\boldsymbol{k}\langle X\rangle$.*
4. *$\Re^\perp$ is a Gröbner basis of the ideal $(\Re^\perp)$ in $\boldsymbol{k}\langle\Xi\rangle$.*

Proof. Let $V = SpanX$, $V^* = Span\ \Xi$.

We show first the implication $1 \Longrightarrow 2$. The implication $2 \Longrightarrow 1$ is analogous.

Suppose condition 1 holds. Clearly, this implies that the algebra $A(\mathbf{k}, X, \Re)$ is a binomial skew polynomial ring. It follows then from Theorem 2.6 that the automorphism $R = R(\Re) : V^2 \longrightarrow V^2$ is a solution of the Yang-Baxter equation. It is not difficult to see that $R^* = R(\Re^*) : (V^*)^2 \longrightarrow (V^*)^2$ is also a solution of the Yang-Baxter equation. Clearly the relations $\Re^*$ are of skew-polynomial type. Hence by theorem 2.6, $\Re^*$ is a Gröbner basis of the ideal $(\Re^*)$ in $\mathbf{k}\langle\Xi\rangle$.

The implication $1 \Longrightarrow 3$ is verified directly by Gröbner bases technique, that is one shows that all ambiguities are solvable, see the Diamond Lemma, [5]. Clearly there are three types of ambiguities: a) $x_kx_jx_i, n \geq k > j > i \geq 1$, b) $x_jx_ix_i, n \geq j > i \geq 1$, and c) $x_jx_jx_i, n \geq j > i \geq 1$. All ambiguities of the type a) are solvable, since by (1) $\Re$ is a Gröbner basis. We will show that all ambiguities of type b) are

solvable. Let j, i be a pair of integers, with $n \geq j > i \geq 1$. Consider the ambiguity $x_j x_i x_i$. It follows from the cyclic condition 1.14 that there exist integers i_1, j_1, j_2, with $1 \leq i_1 < j_1, j_2 \leq n$ such that $\Re$ contains the relations: $x_j x_i - c_{ij} x_{i_1} x_{j_1}$ and $x_{j_1} x_i - c_{ij_1} x_{i_1} x_{j_2}$, where c_{ij} and c_{ij_1} are nonzero coefficients. This gives the following sequence of reductions:

$$x_j x_i x_i \longrightarrow^{R^{12}} (c_{ij} x_{i_1} x_{j_1}) x_i \longrightarrow^{R^{23}} c_{ij} x_{i_1} (c_{ij_1} x_{i_1} x_{j_2}) = c_{ij} c_{ij_1} [x_{i_1} x_{i_1}] x_{j_2} \longrightarrow^{R^{12}} 0.$$

The other possible way of reducing $x_j x_i x_i$ is

$$x_j x_i x_i \longrightarrow^{R^{23}} 0.$$

We have proved that all ambiguities of the type b) are solvable. An analogous argument shows that the ambiguities of the type c) are also solvable. Thus $\Re^!$ is a Gröbner basis of the ideal $(\Re^!)$ in $\mathbf{k}\langle X \rangle$. □

Corollary 3.8. *Let $A = \boldsymbol{k}\langle X \rangle / (\Re)$ be a binomial skew-polynomial ring, let $A^!$ be its Koszul dual. Let $\mathcal{F} = \boldsymbol{k}\langle X \rangle / (\Re^!)$ Then*

(1) *$\mathcal{F}$ has a $\boldsymbol{k}$-basis the set*
$$\mathcal{N}^! = \{x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_n^{\varepsilon_n} \mid \varepsilon_i = 0, 1, \ \textit{for} \ \ \textit{all}\ 1 \leq i \leq n\}.$$
(2) *$A^!$ has a $\boldsymbol{k}$-basis the set*
$$\mathcal{N}(\xi)^! = \{\xi_1^{\varepsilon_1} \xi_2^{\varepsilon_2} \cdots \xi_n^{\varepsilon_n} \mid \varepsilon_i = 0, 1, \ \textit{for} \ \ \textit{all}\ 1 \leq i \leq n\}.$$
(3) *The principal monomial of A has a regular presentation $W = x_1 x_2 \cdots x_n$.*
(4) *The socle of $A^!$ is one dimensional and is generated by the monomial $W(\xi) = \xi_1 \xi_2 \cdots \xi_n$.*

Remark 3.9. The semigroup $\mathcal{S}^! = \langle X; (\Re_0 \bigcup \Re_1) \rangle$ can be presented as $\mathcal{S}^! \simeq \mathcal{S}_0 / (\Re_1)$. It is a semigroup with 0, $xx = 0$ for every $x \in X$. To make the computations in $\mathcal{S}^!$ we compute modulo the relations $\Re_0$, and keep in mind that $\omega \in \langle X \rangle$, is equal to 0 in $\mathcal{S}^!$ if and only if it can be presented as $\omega = \omega'$ in $\mathcal{S}_0$, where $\omega' = axxb \in \langle X \rangle$, for some $x \in X, a, b \in \langle X \rangle$. Denote

$$\mathcal{N}_0^! = \mathcal{N}^! \bigcup \{0\}$$

We can identify $\mathcal{S}^!$ with the semigroup $(\mathcal{N}_0^!, *)$ where the operation $*$ on $\mathcal{N}_0^!$ is defined as follows: for $u, v \in \mathcal{N}_0^!$, either a) $u * v = 0$ and this is true if and only if the normal form $Nor_{\Re_0}(uv)$ contains some subword of the shape $xx, x \in X$, or b) $u * v = w \in \mathcal{N}^!$, where $Nor_{\Re_0}(uv) = w$ (or equivalently $Nor_{\Re}(uv) = cw$, for some nonzero coefficient c).

All relations in $\mathcal{S}_0$, which do not involve subwords of the shape xx are preserved in $\mathcal{S}^!$. In particular, the cyclic conditions are in force.

If $u, v, w \in \mathcal{N}_0$ and $u * w \neq 0$ (that is $u * w \in \mathcal{N}^!$), then each of the equalities $u * w = v * w$ and $w * u = w * v$ implies $u = v$, i.e. $(\mathcal{N}_0^!, *)$ has cancellation low for non-zero products.

Theorem A verifies the Frobenius property for each quadratic algebra with relations of the type $\Re^!$. We prove first some more statements under the hypothesis of Theorem A.

Before proving the theorem we need some more statements.

Clearly the assumption that $A = \mathbf{k}\langle X \rangle / (\Re)$ is a binomial skew-polynomial ring, implies that $\mathcal{S}_0 = \langle X; \Re_0 \rangle$ is a semigroup of skew-polynomial type. (with respect

to the degree-lexicographic order $<$ on $\langle X\rangle$ defined by $x_1 < x_2 < \cdots < x_n$. It is proven in [10], that $\mathcal{S}_0$ satisfies the cyclic condition, therefore Ore condition holds. Furthermore $\mathcal{S}_0$ is with cancellation law, [11]. Proposition 3.10 is true for arbitrary semigroup of skew-polynomial type. In some parts we use argument similar to the proof of Proposition 2.21, but we prefer to give sketch of the proofs explicitly, since they are made under different assumptions.

Proposition 3.10. *Let $\mathcal{S}_0 = \langle X; \Re_0\rangle$ be a semigroup of skew-polynomial type, with respect to the degree-lexicographic ordering $\prec$ on $\langle X\rangle$. Then the following conditions are satisfied.*

1. *The monomial $W_1 = x_1x_2\cdots x_n$ is normal.*
2. *For any $j, 1 \leq j \leq n-1$, there exist a unique $\eta_j \in X$, such that $x_{j+1}\cdots x_n\eta_j = x_jx_{j+1}\cdots x_n$.*
3. *The elements $\eta_1, \cdots, \eta_{n-1}$ are pairwise distinct.*
4. *For every $j, 1 \leq j \leq n-1$, the monomial $W_j = x_jx_{j+1}\cdots x_n$ has exactly $n-j+1$ heads, namely :*
$$H_{W_j} = \{x_j, x_{j+1}, \cdots, x_n\}.$$
5. *For any $j, 1 \leq j \leq n-1$, there exist a unique $\theta_{j+1} \in X$, such that $\theta_{j+1}x_1\cdots x_j = x_1x_2\cdots x_{j+1}$.*
6. *The elements $\theta_2, \cdots, \theta_n$ are pairwise distinct.*
7. *For every $j, 1 \leq j \leq n-1$ the monomial $\omega_j = x_1x_2\cdots x_j$ has exactly j tails, namely*
$$T_{\omega_j} = \{x_1, x_2, \cdots, x_j\}.$$
In particular, every $x_i, 1 \leq i \leq n$ occurs as a head and as a tail of the monomial $W_1 = x_1x_2\cdots x_n = \omega_n$,
8. *The monomial W_1 is the principal monomial of $\mathcal{S}_0$ with a regular presentation $W_1 = x_1x_2\cdots x_n$.*

Under the assumption of Proposition 3.10 we prove first Lemma 3.11. Although the statements of Lemmas 3.11 and 2.22, are analogous, due to the different hypotheses, we need different arguments for their proofs.

Lemma 3.11. *For each integer j, $1 \leq j \leq n-1$, let $\zeta_{j,j+1}, \cdots, \zeta_{j,n}$, $\eta_{j,j+1}, \cdots, \eta_{j,n}$ be the elements of X uniquely determined by the relations*

$$\begin{array}{l}(\zeta_{j,j+1}\eta_{j,j+1} = x_jx_{j+1}) \in \Re_0\\ (\zeta_{j,j+2}\eta_{j,j+2} = \eta_{j,j+1}x_{j+2}) \in \Re_0\\ \cdots\cdots\cdots\\ (\zeta_{j,n-1}\eta_{j,n-1} = \eta_{j,n-2}x_{n-1}) \in \Re_0\\ (\zeta_{j,n}\eta_{j,n} = \eta_{j,n-1}x_n) \in \Re_0.\end{array} \tag{3.3}$$

Then for each j, $1 \leq j \leq n-1$, the following conditions hold:

1. *$\zeta_{j,j+s} \neq \eta_{j,j+s-1}$, for all $s, 2 \leq s \leq n-j$.*
2. *The following are equalities in $\mathcal{S}_0$:*
$$\begin{aligned}\zeta_{j,j+1}\zeta_{j,j+2}\cdots\zeta_{j,n} &= x_{j+1}\cdots x_n\\ x_{j+1}x_{j+2}\cdots x_n\eta_{j,n} &= x_jx_{j+1}\cdots x_n.\end{aligned}$$

(3) *The elements* $\eta_{j,n}, \eta_{j+1,n}, \cdots, \eta_{n-1,n}$ *are pairwise distinct.*

Proof. Condition 1 is obvious. To prove the remaining conditions we use decreasing induction on j, $1 \leq j \leq n-1$.

Step 1. $j = n-1$. Clearly, $x_{n-1}x_n$ is normal thus (cf. Remark 2.15) the relation in $\Re_0$ in which it occurs has the shape $x_{n-1}x_n = \zeta_{n-1,n}\eta_{n-1,n}$, with $\zeta_{n-1,n} \succ x_{n-1}$. It is clear then that $\zeta_{n-1,n} = x_n$ and $x_{n-1}x_n = x_n\eta_{n-1,n}$. Hence the set of heads of $x_{n-1}x_n$ is $\{x_{n-1}, x_n\}$. This gives 2, 3 is trivial.

Step 2. Using decreasing induction on j we first prove condition (3) for all $j, 1 \leq \jmath \leq n-1$. (Step 1, $j = n-1$ gives the base for the induction. Assume that for all $k, n-1 \geq k > j$, the elements $x_k, x_{k+1}, \cdots, x_n$, $\zeta_{k,k+1}, \cdots, \zeta_{k,n}$, $\eta_{k,k+1}, \cdots, \eta_{k,n}$ satisfy

$$\begin{aligned} &(\zeta_{k,k+1}\eta_{k,k+1} = x_k x_{k+1}) \in \Re_0 \\ &(\zeta_{k,k+2}\eta_{k,k+2} = \eta_{k,k+1}x_{k+2}) \in \Re_0 \\ &\cdots\cdots\cdots \\ &(\zeta_{k,n-1}\eta_{k,n-1} = \eta_{k,n-2}x_{n-1}) \in \Re_0 \\ &(\zeta_{k,n}\eta_{k,n} = \eta_{k,n-1}x_n) \in \Re_0; \end{aligned} \tag{3.4}$$

all $\eta_{j+1,n}, \eta_{j+2,n}, \cdots, \eta_{n-1,n}$ are pairwise distinct, and the modified conditions (3), in which "j" is replaced by "k" hold. Let $\zeta_{j,j+1}, \cdots \zeta_{j,n}$, $\eta_{j,j+1} \cdots \eta_{j,n}$ satisfy (3.4). We shall prove that $\eta_{j,n} \neq \eta_{k,n}$, for all $k, j < k \neq n-1$. Assume the contrary,

$$\eta_{j,n} = \eta_{k,n} \tag{3.5}$$

for some $k > j$. It follows from (3.5), the relations

$$\xi_{j,n}\eta_{j,n} = \eta_{j,n-1}y_n, \text{ and } \xi_{k,n}\eta_{k,n} = \eta_{k,n-1}y_n,$$

and the Ore condition that

$$\eta_{j,n-1} = \eta_{k,n-1}.$$

Similar argument implies in $n-k$ steps the equalities

$$\eta_{j,n} = \eta_{k,n}, \quad \eta_{j,n-1} = \eta_{k,n-1}, \quad \cdots, \quad \eta_{j,k+1} = \eta_{k,k+1}.$$

Now the relations

$$\zeta_{j,k+1}\eta_{j,k+1} = \eta_{j,k}x_{k+1}, \quad \zeta_{k,k+1}\eta_{k,k+1} = x_k x_{k+1}$$

and the Ore condition again imply $\eta_{j,k} = x_k$. By (3.3) we have

$$(\zeta_{j,k}\eta_{j,k} = \eta_{j,k}x_k) \in \Re_0$$

This is impossible, since $\eta_{j,k} = x_k$, and the relations in $\Re_0$ are square-free.

We have shown that the assumption $\eta_{j,n} = \eta_{k,n}$, for some $k > j$, leads to a contradiction. This proves (3) for all $j, 1 \leq j \leq n-1$.

We set

$$\eta_1 = \eta_{1,n}, \quad \eta_2 = \eta_{2,n}, \ \cdots, \ \eta_{n-1} = \eta_{n-1,n}.$$

Next we prove (2).

By the inductive assumption we have

$$\zeta_{k,k+1}\zeta_{k,k+2}\cdots\zeta_{k,n} = x_{k+1}\cdots x_n \in \mathcal{N}_0$$

and

$$x_{k+1}\cdots x_n.\eta_{k+1} = x_k\cdots x_n.$$

Applying the relations 3.3 we obtain

$$(\zeta_{j,j+1}\zeta_{j,j+2}\cdots\zeta_{j,n})\eta_j = x_j x_{j+1}\cdots x_n \in \mathcal{N}_0.$$

Denote the normal form $Nor_{\Re_0}(\zeta_{j,j+1}\zeta_{j,j+2}\cdots\zeta_{j,n})$ modulo the Gröbner basis $\Re_0$, as

$$\upsilon_j = Nor_{\Re_0}(\zeta_{j,j+1}\zeta_{j,j+2}\cdots\zeta_{j,n}),$$

clearly, $\upsilon_j \in \mathcal{N}_0$.

We have to show that there is an equality of words in $\langle X\rangle$.

$$\upsilon_j = x_{j+1}x_{j+2}\cdots x_n.$$

The equality,

$$\upsilon_j\eta_j, = x_j x_{j+1}\cdots x_n \in \mathcal{N}$$

implies

$$Nor(\upsilon_j\eta_j) = x_j x_{j+1}\cdots x_n,$$

as words in the free semigroup $\langle X\rangle$. Furthermore υ_j does not contain subwords of the type xx, (this can be easily seen using the weak cyclic condition). Thus

$$\upsilon_j = x_{j_1}x_{j_2}\cdots x_{j_{n-1}}, \text{ where} j_1 < j_2 \cdots < j_{n-1} \leq n.$$

and therefore

$$j_1 \leq j+1. \tag{3.6}$$

The theory of Gröbner basis implies the following relations in $\langle X\rangle$.

$$x_j x_{j+1}\cdots x_n = Nor(\upsilon_j\eta_j) \preceq \upsilon_j\eta_j = x_{j_1}x_{j_2}\cdots x_{j_{n-1}}\eta_j,$$

therefore $j \leq j_1$. By the last inequality, and (3.6) only two cases are possible: **a.** $j_1 = j$; and **b.** $j_1 = j+1$. Assume that $j_1 = j$. It follows then that

$$\upsilon_j = x_j\cdots x_{k-1}x_{k+1}\cdots x_n,$$

for some $k, k \geq j+1$ (In the case when $k = j+1$, $\upsilon_j = x_j x_{j+2}\cdots x_n$). Thus the equalities

$$\upsilon_j\eta_k = x_j\cdots x_{k-1}(x_{k+1}\cdots x_n\eta_k) = x_j x_{j+1}\cdots x_n = \upsilon_j\eta_j,$$

hold in $\mathcal{S}_0$. So, by the cancellation low in $\mathcal{S}_0$, we obtain $\eta_k = \eta_j$, with $j < k$, which is impossible. Hence the assumption $j_1 = j$ leads to a contradiction. This verifies $j_1 = j+1$, which implies $\upsilon_j = x_{j+1}\cdots x_n$, and therefore the desired equality

$$x_{j+1}\cdots x_n\eta_j = x_j\cdots x_n$$

holds in $\mathcal{S}_0$. The lemma has been proved. □

Proof of Proposition 3.10. Condition (1) is obvious. Lemma 3.11 proves 2, 3. By the choice of $\eta_i,\ 1 \leq i \leq n-1$, the following equalities hold in $\mathcal{S}_0$:

$$\begin{aligned} x_n\eta_{n-1}\eta_{n-2}\cdots\eta_j &= x_{n-1}x_n\eta_{n-2}\cdots\eta_j \\ &= \cdots\cdots\cdots\cdots\cdots \\ &= x_{j+1}\cdots x_{n-1}x_n\eta_j \\ &= x_{j+1}\cdots x_{n-1}x_n, \end{aligned}$$

which implies (4). The proof of conditions (5), (6), and (7) is analogous to the proof of (2), (3), and (4), respectively. It follows from the weak cyclic condition, that the normal form $Nor(u)$ of a monomial $u \in \langle X\rangle$, with the shape $u = ayyb, y \in X$ has the shape $Nor(u) = a_1xxb_1 \in \mathcal{N}_0, x \in X$. Therefore W is the principal monomial of $\mathcal{S}_0$. Condition (8) is obvious.

The following lemma is used for the Frobenius property.

Lemma 3.12. *For any monomial $u \in \mathcal{N}^!$ there exist uniquely determined u' and u'' in $\mathcal{N}^!$, such that*

$$u * u' = x_1 x_2 \dots x_n, \quad u'' * u = x_1 x_2 \dots x_n. \tag{3.7}$$

Proof. Let u be an element of $\mathcal{N}^!$. Then

$$u = x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n}$$

where for all i, $1 \leq i \leq n$, one has $0 \leq \varepsilon_i \leq 1$. Let $\eta_i, \theta_j,\ 1 \leq i, j-1 \leq n-1,$ be as in Proposition 3.10. Let

$$\begin{aligned} u' &= x_n^{(1-\varepsilon_n)} * \eta_{n-1}^{(1-\varepsilon_{n-1})} * \cdots * \eta_1^{(1-\varepsilon_1)}. \\ u'' &= \theta_n^{(1-\varepsilon_n)} * \theta_{n-1}^{(1-\varepsilon_{n-1})} * \cdots * \theta_2^{(1-\varepsilon_2)} x_1^{(1-\varepsilon_1)}. \end{aligned}$$

It is easy to verify that the equalities 3.7 hold. The uniqueness of u' and u'' follows from the cancellation law in $\mathcal{S}_0$. □

Proof of Theorem A. Let $\mathcal{F}$ be the quadratic algebra from the hypothesis of Theorem A.

Then Lemma 3.7 and Corollary 3.8 imply conditions (1) and (2) of the theorem. For $0 \leq i$ we set

$$\begin{aligned} \mathcal{N}_i^! &= \{u \in \mathcal{N}^! \mid u \text{ has length } i\} \\ \mathcal{F}_i &= Span_{\mathbf{k}}\, \mathcal{N}_i^!. \end{aligned} \tag{3.8}$$

It is clear that $\mathcal{F}_0 = \mathbf{k}$, $\mathcal{F}_i = 0$, for $i > n$ and for $1 \leq i \leq n$ one has

$$dim_{\mathbf{k}}\ \mathcal{F}_i = \sharp\, \mathcal{N}_i^! = \binom{n}{i},$$
$$\text{in particular, } dim_{\mathbf{k}} \mathcal{F}_n = 1.$$

Clearly, $\mathcal{F}$ is graded : $\mathcal{F} = \bigoplus_{0 \leq i \leq n} \mathcal{F}_i$, $\mathcal{F}_i = 0$, for $i > n$.

It follows from Lemma 3.12 that the map

$$(-,-) : \mathcal{F}_i \times \mathcal{F}_{n-i} \to \mathcal{F}_n$$

defined by

$$(u, v) := \text{ the normal form of } uv \text{ in } \mathcal{F}$$

is a perfect duality. This proves Theorem A.

Now we can prove Theorem 3.1

Proof of Theorem 3.1. Let A be a binomial skew polynomial ring. By Fact 2.5 every algebra with quadratic Gröbner basis is Koszul, this implies the Koszulity of A. Furthermore from [1] one deduces that for every graded $\mathbf{k}$-algebra $\mathcal{B}$ with quadratic Gröbner basis, Anick's resolution of $\mathbf{k}$ as a $\mathcal{B}$-module is minimal. We shall use now the terminology of Anick. The set of obstructions (i.e. the leading monomials of the elements of the reduced Gröbner basis) for a binomial skew polynomial A is $\{x_j x_i \mid 1 \leq i < j \leq n\}$. Therefore the maximal k for which there exist k-chains is $k = n-1$. In fact the only $n-1$-chain is $x_n x_{n-1} \cdots x_1$. It follows then from a theorem of Anick, [1], that $gl.dim A = n$. We have shown that A is a Koszul algebra of finite global dimension. Clearly, A has polynomial growth. Furthermore, by Theorem A, the Koszul dual $A^!$ is Frobenius. It follows then from 3.2 that A is Gorenstein, and therefore A is Artin-Schelter regular.

Proof of Theorem B. Let $A = \mathbf{k}\langle X\rangle/(\Re)$ be a quantum binomial algebra. The implication $1 \Longrightarrow 2$ follows from Theorem 2.20. Assume now that A is binomial skew polynomial ring. By remark 3.3 A satisfies the cyclic condition (see also [10]), and therefore it satisfies the weak cyclic condition. By Corollary 3.8 the Koszul dual $A^!$ is Frobenius, and has regular socle. This proves the implication $2 \Longrightarrow 1$.

The equivalence of conditions (2) and (3) follows from Theorem 2.6 (see also [14] Theorem 9.7).

We have shown that conditions (1), (2), and (3) are equivalent.

Now it is enough to show that every binomial skew polynomial ring A satisfies the conditions (a) , ..., (e). Conditions (a) and (b) are clear. We have shown that A is Artin Schelter regular. It is shown in [16], Corollary 1.6 that A is a domain. It is proven in [10] (see also [9], and [16]) that A is left and right Noetherian. It follows from [15] that A satisfies polynomial identity. Now as a finitely generated PI algebra, A is catenary, see [27].

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